

LKTG: Games on Changing Graphs

Complete statement edition · version 7, August 11, 2026

Presented by Maksim Didin, Mark Pimenov, Matvey Surkov, and ChatGPT.

This project is devoted to games in which the graph changes during play. Players move through previously unvisited vertices, orient and color edges, and, in the final problems, construct graphs with prescribed game-theoretic properties. The first problems introduce the main ideas; the hardest ones lead to major theorems and five open items.

How the project works

You may work individually or in a fixed team. Team membership is settled before the project begins and cannot change before the project ends. No single final report is required. Any numbered problem or lettered part may be submitted orally or in writing at any time while the project is running, up to the announced deadline. After receiving comments, you may submit a corrected solution. Each part may be submitted separately.

A solution must contain a complete proof and account for every legal response by the opponent. Checking small cases alone does not prove a general claim. Every book, article, or program used in a solution must be cited, and arguments obtained independently must be clearly distinguished from results obtained with their help.

Stars. Exactly five open items are marked with ★: 11(g), 11(h), 13(d), 14(f), and Problem 27. For these problems, you may submit proved special cases, new lemmas, or verified computational results. AI may be used only on parts explicitly marked with ★; it is forbidden everywhere else in the project. A submission for a starred problem must describe exactly how AI was used and clearly separate the participants' own proofs from statements obtained computationally or with AI assistance.

Conventions. Unless stated otherwise, all graphs are finite and simple, and the length of a path or cycle is its number of edges. Every game has perfect information and no random moves. Passing is forbidden unless explicitly allowed. If an immediate winning condition is specified, the game ends as soon as that condition is first met. A draw is possible only when it is explicitly mentioned. Difficulty is rated from $1/5$ to $5/5$ for strong high-school students: $1/5$ is a warm-up; $2/5$ is roughly an hour's work; $3/5$ is a substantial one-session problem; $4/5$ may take several hours; and $5/5$ is a culminating problem or a mini-project.

Route

- A.** Unvisited vertices and adding edges (1–4).
- B.** Dynamic orientations (5–9).
- C.** Edge-coloring and clique games (10–15).
- D.** Graphs for games (16–17).
- E.** One-objective weight strategies (18–23).
- F.** Two-objective weight strategies (24–27).
- G.** Dynamics and tournaments (28–29).

A. Unvisited vertices and new edges

1. A game with no returns

Difficulty: a) 3/5; b) 2/5; c) 3/5; d) 2/5 · Source: a) Fraenkel-Scheinerman-Ullman, 1993; b), c) variants by the project editors; d) suggested by a coauthor

In a finite graph G , Petya chooses a vertex s , places a token there, and declares it visited. Petya and Vasya then alternate moving the token along an edge to an unvisited vertex; Vasya makes the first move. A player who cannot move loses.

- Prove that Vasya wins if and only if G has a perfect matching, that is, a matching covering every vertex. In particular, Petya wins whenever $|V(G)|$ is odd.
- Change the rules as follows: once during the game, each player may pass instead of moving the token. A pass is allowed even when no move is available. Prove that the criterion in part a) remains unchanged.
- Petya now chooses two distinct vertices s and t , places a token at each of them, and declares both vertices visited. On each turn, a player chooses one token and moves it along an edge to an unvisited vertex; Vasya moves first. For every even $n \geq 4$, construct a connected graph G on n vertices with a perfect matching and give two choices of $\{s, t\}$, one for which Vasya wins and one for which Petya wins.
- Suppose that G has neither a perfect matching nor an isolated vertex. Prove that Petya wins for at least two distinct choices of the initial vertex s .

2. Coins on a graph

Difficulty: a) 2/5; b) 4/5; c) 2/5; d) 3/5 · Source: suggested by a coauthor; c), d) by the project editors

Each vertex of a finite connected graph G contains a nonnegative integer number of coins. Petya chooses a starting vertex, places a token there, declares the vertex visited, and takes all its coins. Vasya then moves first. On each move, the token must travel along an edge to an unvisited vertex, and the player takes all coins from the new vertex. The game ends when the next player cannot move. The player with more coins wins; equal totals give a draw.

- Suppose that G is a path. Prove that Petya can choose the starting vertex and then play so as not to lose.
- Prove the same statement when G is an arbitrary finite tree.
- Prove the same statement when G is a cycle.
- Is the statement true for every finite connected graph G ?

3. Adding edges until the first odd cycle

Difficulty: a) 3/5; b) 5/5; c) 3/5 · Source: a), b) Canadian Mathematical Olympiad 2019, Problem 5; c) suggested by a coauthor

Start with a fixed set of $n \geq 3$ vertices and no edges. Petya and Vasya alternately add one missing edge, with Petya moving first. The player whose move first creates an odd cycle loses immediately.

- Prove that Vasya wins when n is odd.
- Determine all even n for which Petya wins.
- Change the rules as follows: after Petya's first move, an edge may not join two isolated vertices. Prove that Petya wins for every even n .

4. Adding edges until the graph is connected

Difficulty: 4/5 · Source: suggested by a coauthor

Start with a fixed set of $n \geq 3$ vertices and no edges. Petya and Vasya alternately add one missing edge, with Petya moving first. The player whose move first makes the graph connected loses immediately. Determine the winner for every $n \geq 3$.

B. Dynamic orientations

5. A token and edge reversals

Difficulty: a) 2/5; b) 4/5 · Source: Tournament of Towns 2018/19

Let G be a connected undirected graph with at least two vertices. Petya first orients every edge and chooses the initial vertex of a token. The game then proceeds in rounds. In each round, Petya must move the token along an outgoing edge, after which Vasya reverses exactly one edge incident with the new vertex; he may reverse the edge just traversed. If the token has no outgoing edge before Petya's move, Vasya wins. If the game continues forever, Petya wins.

- a) If G is a tree, prove that Petya can choose an initial position and keep moving forever against every response by Vasya.
- b) If G contains a cycle, prove that Vasya wins after finitely many rounds, regardless of the initial position and Petya's moves.

6. Two tokens and reversing arrows

Difficulty: a) 2/5; b) 4/5 · Source: M. Didin, 2026; part a) suggested by a coauthor

Two distinct vertices A and B are chosen in a finite oriented graph. Petya's token starts at A , Vasya's token at B , and there is a directed path from A to B . Whenever a token moves, it follows an outgoing arrow, and the traversed edge is immediately reversed. Both tokens may occupy the same vertex; the position of one token does not restrict the movement of the other. Petya wants to bring his token to B .

- a) Vasya first makes any finite number of legal moves with his token, possibly none, and then stops moving. From then on, only Petya moves. Prove that Petya can reach B , regardless of Vasya's moves.
- b) The players now alternate turns, with Petya moving first. On a turn, a player may either pass or move their token. Petya wins as soon as his token first reaches B ; if this never happens, Vasya wins. Is there an initial directed graph on which Vasya can prevent Petya from winning?

Common rules for Problems 7–9. On an undirected graph G , Cycle-Maker and Cycle-Breaker alternately choose an unoriented edge and direct it either way. A directed cycle may contain arrows oriented by both players. Cycle-Maker wins immediately when a directed cycle of length at least 3 appears; if every edge is oriented and there is no such cycle, Cycle-Breaker wins. On an infinite graph, Cycle-Maker must create a finite directed cycle in finitely many moves; if this never happens, Cycle-Breaker wins.

7. 2-degenerate graphs

Difficulty: a) 2/5; b) 4/5 · Source: problem by the project editors

A graph is called 2-degenerate if each of its nonempty subgraphs has a vertex of degree at most 2.

- a) Prove that a graph is 2-degenerate if and only if its edges have an acyclic orientation in which every vertex has in-degree at most 2. No game is played in this part: the task is simply to orient the edges of the graph.
- b) Let G be a finite 2-degenerate graph. Prove that Cycle-Breaker wins if $|E(G)|$ is even and Cycle-Maker moves first, and also if $|E(G)|$ is odd and Cycle-Breaker moves first.

8. Square grids

Difficulty: a) 4/5; b) 3/5 · Source: the orientation game on grids

- a) Consider the infinite square grid with vertex set $\mathbb{Z}^2$, where vertices at Euclidean distance 1 are adjacent. Determine the winner when Cycle-Maker moves first and when Cycle-Breaker moves first.
- b) Let $m, n \geq 1$, and let $P_m \square P_n$ be the finite rectangular grid, where P_k denotes the path on k vertices. Prove that Cycle-Breaker wins if m and n have the same parity and Cycle-Maker moves first, and also if m and n have opposite parity and Cycle-Breaker moves first.

9. The Bollobás-Szabó game

Difficulty: a) 4/5; b) 4/5; c) 5/5; d) 2/5; e) 2/5; f) 2/5 · Source: Bollobás-Szabó, 1998

A sink is a vertex with no outgoing arrow, and a source is a vertex with no incoming arrow.

Write $v(H)$ and $e(H)$ for the numbers of vertices and edges of a graph H . Call a finite simple graph H a minimal dense core if

$$v(H) \geq 3, \quad e(H) = 2v(H) - 2,$$

and every proper subgraph $F \subsetneq H$ with $v(F) \geq 3$ satisfies $e(F) \leq 2v(F) - 3$.

- a) Prove that Cycle-Maker has a winning strategy when moving first on any finite simple 4-regular graph.
- b) The game is played on a minimal dense core H . Cycle-Breaker moves first, and Cycle-Maker moves second. In this part the players orient every edge of H , even if a directed cycle has already appeared. Prove that Cycle-Maker can ensure that the final orientation of H has no sinks.
- c) Cycle-Maker moves first on a minimal dense core H . Prove that he can play so that either a directed cycle appears, or the final orientation after all edges have been directed has no sinks or has no sources.
- d) Deduce the Bollobás-Szabó theorem: if a finite simple graph G has $N \geq 3$ vertices and at least $2N - 2$ edges, then Cycle-Maker has a winning strategy when moving first.

e) Prove that Cycle-Maker wins on every finite simple 4-regular graph, regardless of who moves first. Now let a simple graph with N vertices be cellularly embedded in a closed connected orientable surface of genus $g \geq 1$, with the boundary walk of every face having length 4. Find the number of edges and prove that Cycle-Maker wins regardless of who moves first.

f) A rooted ball of radius R is the subgraph induced by the vertices at distance at most R from a distinguished center. For every $R \geq 0$, construct a finite simple graph G_R and a vertex v_R such that the rooted ball of radius R centered at v_R is isomorphic to the rooted ball of radius R in the square grid, but for either choice of the first player the winner on G_R is the opposite of the winner in the corresponding game on the infinite grid. Give all parameters of the construction explicitly.

Original source: B. Bollobás, T. Szabó, The oriented cycle game, Discrete Mathematics 186 (1998), 55–67.

C. Edge-coloring and clique games

For a graph H , let $\omega(H)$ be its clique number and let $\alpha(H)$ be its independence number.

10. The first monochromatic triangle

Difficulty: 2/5 · Source: Simon Marais Mathematics Competition 2017, A1; generalized here

Let $n \geq 5$. Blue and Red alternately color uncolored edges of K_n in their own colors, with Blue moving first. A player wins immediately when three edges of that player's color form a triangle for the first time. If every edge has been colored and neither player has such a triangle, the game is a draw. Determine the outcome under optimal play.

11. Erdős's symmetric game

Difficulty: a) 2/5; b) 2/5; c) 2/5; d) 3/5; e) 3/5; f) 5/5; g★) open; h★) open · Source: Erdős; Malekshahian-Spiro, 2026; Cambie-Provoost, 2025

Let $n \geq 3$. Red and Blue alternately choose an uncolored edge of K_n and color it in their own color, with Red moving first. After every edge has been colored, Red wins if his largest clique is strictly larger than Blue's; if it is no larger, Blue wins. Let R and B denote the graphs formed by the red and blue edges, respectively.

a) Prove that Red can always ensure $\omega(R) \geq \omega(B)$.

b) Prove that Blue can always ensure

$$\omega(R) \leq \omega(B) + 1.$$

c) Replace the complete graph by an arbitrary finite graph G with at least one edge. Red and Blue alternately color its uncolored edges, with Red moving first. Prove that Red can ensure $\omega(R) \geq \omega(B)$, while Blue can ensure $\omega(R) \leq \omega(B) + 1$. Thus, under optimal play, the difference of the two clique numbers is either 0 or 1.

d) Suppose that Red can force a strict win on K_n . Prove that Blue has a winning strategy on K_{n+1} .

e) Suppose that Red can force a strict win on K_n . Prove that Blue has a winning strategy on K_{n+2} .

f) Let $n > 100$, and suppose that Red can force a strict win on K_n . Prove that Blue has a winning strategy on K_{n+3} .

g★) Prove or disprove: there is a positive integer N such that, for every $n \geq N$, if Red can force a strict win on K_n , then Blue has a winning strategy on K_{n+4} .

h★) Prove or disprove: Blue wins under optimal play for every $n \geq 3$.

12. The biased 1 : 2 game

Difficulty: a) 1/5; b) 2/5 · Source: problem by the project editors

Let $n \geq 3$. Bella and Chingiz play on the edges of the complete graph K_n , with Bella moving first. In each round Bella colors one uncolored edge white, after which Chingiz colors two uncolored edges black, or all remaining edges if fewer than two are left. Once every edge has been colored, Chingiz wins if the largest black clique is strictly larger than the largest white clique; otherwise Bella wins.

a) If W is the graph of white edges, explain why $\omega(\text{black graph}) = \alpha(W)$.

b) For $n = 4$, give an explicit winning strategy for Chingiz.

13. Bella and Chingiz on K_n

Difficulty: a) 3/5; b) 4/5; c) 5/5; d★) open · Source: YUMT 2025 Grand Final, Problem 3; Erdős; Cambie-Provoost, 2025

Let $n \geq 3$ and $k \geq 1$ be integers. Bella and Chingiz play on the edges of K_n , with Bella moving first. In each round Bella colors one uncolored edge white. If uncolored edges remain, Chingiz colors k of them black, or all of them if fewer than k remain. The game ends as soon as the last edge is colored. Chingiz wins if and only if the largest black clique is strictly larger than the largest white clique.

a) Let $n > 3$ be even and let $k = 4$. Give an explicit winning strategy for Chingiz.

b) Let $n > 3$ be even and let $k = 3$. Give an explicit winning strategy for Chingiz.

c) Let $n > 3$ be odd and let $k = 3$. Give an explicit winning strategy for Chingiz.

d★) Prove or disprove: for $k = 2$, Chingiz has a winning strategy for every even $n > 3$.

14. A white clique on an infinite board

Difficulty: a) 3/5; b) 3/5; c) 4/5; d) 5/5; e) 5/5; f★) open · Source: a) Pekeč, 1996; b) by the project editors; c), d) Beck, 2002, Pekeč, 1996; e) Beck, 2002; f) Beck, 2002, Gebauer, 2012

Let $b \geq 1$ and $r \geq 2$ be integers. Bella and Chingiz play on the edges of the countably infinite complete graph K_∞ . In each round Bella colors one uncolored edge white, after which Chingiz colors b uncolored edges black. Bella wins as soon as a white clique K_r appears. Let $T_b(r)$ be the minimum number of Bella moves needed to guarantee a win.

a) Prove that Bella has a winning strategy for all integers $b \geq 1$ and $r \geq 2$; in particular, $T_b(r)$ is finite.

b) Prove that

$$T_b(3) = b + 3.$$

c) For $r \geq 17$, prove that

$$\lfloor 2^{r/2} \rfloor \leq T_1(r) \leq (r-3)2^{r-1} + r + 1.$$

d) For $r \geq 30$, prove that

$$\lfloor 3^{r/2} \rfloor \leq T_2(r) \leq \sum_{j=0}^{r-2} j3^j + r + 1 < r3^{r-1}.$$

e) Prove Beck's bound

$$T_1(r) < 2^{r+2}.$$

f★) For large r , narrow the gap between the best known bounds

$$2^{r/2} \leq T_1(r) \leq 2r^7 2^{2r/3}.$$

Comment. The upper bound in part f) is due to Gebauer and is substantially stronger than Beck's bound, but the known proof is too technical for this project.

15. Relating the 1 : 2 and 1 : 1 games

Difficulty: 3/5 · Author: ChatGPT, 2026

Let $N \geq r \geq 2$. On the edges of K_N , Bella colors one uncolored edge white in each round, after which Chingiz colors two uncolored edges black; if fewer than two uncolored edges remain, he colors all of them. Bella wins as soon as a white clique K_r appears. If every edge is colored without a white K_r appearing, Chingiz wins. Suppose that Bella has a winning strategy in this game.

Prove that, under the same rules except that Chingiz colors only one black edge per round, Bella can force a white clique K_{r+1} on every complete graph K_M with

$$M \geq 6N + 1.$$

D. Graphs for games

16. A 2-connected board for Red

Difficulty: 4/5 · Source: Malekshahian-Spiro, 2026

Red and Blue alternately color uncolored edges of a fixed finite graph H , with Red moving first. After every edge has been colored, Red wins if his largest clique is strictly larger than Blue's; otherwise Blue wins.

For every positive integer M , construct a 2-connected graph H with at least M vertices on which Red has a winning strategy. Here 2-connected means that the graph is connected and remains connected after the deletion of any one vertex.

17. Companies and geniuses

Difficulty: a) 2/5; b) 3/5; c) 4/5 · Source: Moscow Mathematical Olympiad 2011; Tournament of Towns 2011; generalization

The vertices of a finite simple graph are programmers, and edges represent acquaintances. Petya's and Vasya's companies alternately hire one programmer who has not yet been hired by either company; Petya's company moves first. Each company may choose its first programmer arbitrarily. Every later hire must know at least one programmer already working for that company. If a company cannot move, it stops hiring, while the other continues under the same rule for as long as possible. Exactly r programmers have been designated as geniuses.

- a) For $r = 3$, construct a graph in which Vasya's company can guarantee hiring at least two geniuses.
- b) For $r = 4$, construct a graph in which Vasya's company can guarantee hiring at least three geniuses.
- c) For every $r \geq 2$, construct a graph in which Vasya's company can guarantee hiring at least $r - 1$ geniuses.

E. One-objective weight strategies

Common rules for Problems 18–23. Let $p, q \geq 1$ be integers. The game is played in rounds on the edges of K_n . In each round Chingiz first colors p uncolored edges black, after which Bella colors q uncolored edges white. If fewer uncolored edges remain than a player is allowed to color, that player colors all remaining edges. Let B and W be the final graphs of black and white edges. In parts concerning the 1 : 1 game, $p = q = 1$. An unspecified logarithm is natural.

18. White cliques

Difficulty: a) 2/5; b) 4/5; c) 1/5 · Source: the Erdős–Selfridge weight method; generalized here

For $2 \leq k \leq n$, let N_k be the number of k -vertex sets for which every internal edge is white at the end of the game.

a) Prove that in the 1 : 1 game Chingiz can guarantee

$$N_k \leq \left\lfloor \binom{n}{k} 2^{-\binom{k}{2}} \right\rfloor.$$

b) Prove that in the $p : q$ game Chingiz can guarantee

$$N_k \leq \left\lfloor \binom{n}{k} (p+1)^{-\binom{k}{2}/q} \right\rfloor.$$

c) Deduce, respectively, that

$$\omega(W) \leq \lceil 2 \log_2 n \rceil + 1 \quad \text{and} \quad \omega(W) \leq \lceil 2q \log_{p+1} n \rceil + 1.$$

19. White Hamiltonian cycles

Difficulty: a) 2/5; b) 3/5; c) 1/5 · Source: problem by the project editors; Hamiltonian game formulation: Chvátal–Erdős, 1978

Let $n \geq 3$. Write $h(W)$ for the number of Hamiltonian cycles in W ; cycles that differ only in their starting vertex or direction of traversal are considered identical.

a) Prove that in the 1 : 1 game Chingiz can guarantee

$$h(W) \leq \left\lfloor \frac{(n-1)!}{2^{n-1}} \right\rfloor.$$

b) Prove that in the $p : q$ game Chingiz can guarantee

$$h(W) \leq \left\lfloor \frac{(n-1)!}{2(p+1)^{n/q}} \right\rfloor.$$

c) Deduce that if

$$(p+1)^{n/q} > \frac{(n-1)!}{2},$$

then Chingiz can prevent a white Hamiltonian cycle.

20. A family of white targets

Difficulty: a) 2/5; b) 4/5 · Source: Erdős–Selfridge, 1973; generalized here

Let $\mathcal{F}$ be an arbitrary family of nonempty sets of edges of K_n . Let $N_{\mathcal{F}}$ be the number of sets $F \in \mathcal{F}$ whose every edge is white at the end of the game.

a) Prove that in the 1 : 1 game Chingiz can guarantee

$$N_{\mathcal{F}} \leq \left\lfloor \sum_{F \in \mathcal{F}} 2^{-|F|} \right\rfloor.$$

b) Prove that in the $p : q$ game Chingiz can guarantee

$$N_{\mathcal{F}} \leq \left\lfloor \sum_{F \in \mathcal{F}} (p+1)^{-|F|/q} \right\rfloor.$$

21. The black degree of every vertex

Difficulty: a) 3/5; b) 4/5 · Source: the Erdős–Selfridge weight method; problem by the project editors

Let $\delta(G)$ denote the minimum degree of a graph G .

a) Prove that for every $\varepsilon > 0$ and all sufficiently large n , Chingiz can guarantee in the 1 : 1 game that

$$\delta(B) \geq \left(\frac{1}{2} - \varepsilon \right) (n-1).$$

b) Prove that for any fixed $p, q \geq 1$ and $\varepsilon > 0$, and for all sufficiently large n , Chingiz can guarantee

$$\delta(B) \geq \left(\frac{p}{p+q} - \varepsilon \right) (n-1).$$

22. Density inside large sets

Difficulty: a) 4/5; b) 5/5 · Source: the Erdős-Selfridge weight method; problem by the project editors

Let $e_B(U)$ denote the number of black edges with both endpoints in U .

a) Prove that for every $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that, for all sufficiently large n , Chingiz can guarantee in the 1 : 1 game that

$$e_B(U) \geq \left(\frac{1}{2} - \varepsilon\right) \binom{|U|}{2}$$

for every set U containing at least $\lceil C \log n \rceil$ vertices.

b) Prove that for any fixed $p, q \geq 1$ and $\varepsilon > 0$, there is a constant $C = C(p, q, \varepsilon)$ such that, for all sufficiently large n , Chingiz can guarantee

$$e_B(U) \geq \left(\frac{p}{p+q} - \varepsilon\right) \binom{|U|}{2}$$

for every set U containing at least $\lceil C \log n \rceil$ vertices.

23. Density between large sets

Difficulty: a) 4/5; b) 5/5 · Source: the Erdős-Selfridge weight method; problem by the project editors

For disjoint vertex sets X and Y , let $e_B(X, Y)$ denote the number of black edges joining X to Y .

a) Prove that for every $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that, for all sufficiently large n , Chingiz can guarantee in the 1 : 1 game that

$$e_B(X, Y) \geq \left(\frac{1}{2} - \varepsilon\right) |X| |Y|$$

for all disjoint sets X, Y , each containing at least $\lceil C \log n \rceil$ vertices.

b) Prove that for any fixed $p, q \geq 1$ and $\varepsilon > 0$, there is a constant $C = C(p, q, \varepsilon)$ such that, for all sufficiently large n , Chingiz can guarantee

$$e_B(X, Y) \geq \left(\frac{p}{p+q} - \varepsilon\right) |X| |Y|$$

for all disjoint sets X, Y , each containing at least $\lceil C \log n \rceil$ vertices.

F. Two-objective weight strategies

24. A white clique and black degrees

Difficulty: a) 3/5; b) 4/5 · Source: problem by the project editors

In the $p : 1$ game, Chingiz first colors p uncolored edges black in each round, after which Bella colors one uncolored edge white. If there are not enough uncolored edges, the player colors all remaining edges. Let B and W be the final graphs.

a) Prove that for every $\varepsilon > 0$ and all sufficiently large n , Chingiz can use one strategy to guarantee simultaneously in the $1 : 1$ game that

$$\omega(W) \leq \lceil 2 \log_2 n \rceil + 1 \quad \text{and} \quad \delta(B) \geq \left(\frac{1}{2} - \varepsilon \right) (n - 1).$$

b) Prove that for any fixed $p \geq 1$ and $\varepsilon > 0$, and all sufficiently large n , Chingiz can use one strategy to guarantee simultaneously in the $p : 1$ game that

$$\omega(W) \leq \lceil 2 \log_{p+1} n \rceil + 1 \quad \text{and} \quad \delta(B) \geq \left(\frac{p}{p+1} - \varepsilon \right) (n - 1).$$

25. Two bounds in the Erdős game

Difficulty: a) 5/5; b) 2/5 · Source: M. Didin, M. Pimenov, 2026; project version

Fix a real number

$$0 < \beta < \frac{2}{3}.$$

Bella and Chingiz play on the edges of K_n . Bella moves first and colors one uncolored edge white on every move; after each of her moves, Chingiz colors two uncolored edges black. If fewer than two uncolored edges remain before a Chingiz move, he colors all remaining edges.

a) Prove that there are a constant $C = C(\beta)$ and an integer n_0 such that, for every $n \geq n_0$, Chingiz can use one strategy to guarantee simultaneously that

$$\omega(W) \leq \lceil 2 \log_3 n \rceil + 1$$

and

$$e_B(U) \geq \beta \binom{|U|}{2}$$

for every set U containing at least $\lceil C \log n \rceil$ vertices.

b) Deduce that for all sufficiently large n ,

$$\omega(B) > \omega(W).$$

26. The $p : q$ game and the golden ratio

Difficulty: 5/5 · Source: M. Didin, M. Pimenov, 2026; generalized here

Let

$$\varphi = \frac{1 + \sqrt{5}}{2}.$$

Fix integers $p, q \geq 1$ satisfying

$$p > \varphi, \quad p + 1 > \left(1 + \frac{q}{p} \right)^{2q}.$$

Bella and Chingiz play on the edges of K_n . Bella moves first and colors q uncolored edges white on every move; after each of her moves, Chingiz colors p uncolored edges black. If fewer uncolored edges remain than a player is allowed to color, that player colors all remaining edges.

Prove that for all sufficiently large n , Chingiz can guarantee

$$\omega(B) > \omega(W).$$

27★. The exact Erdős conjecture

Difficulty: open problem · Source: Erdős; R. K. Guy, 1983

Bella and Chingiz play on the edges of K_n . Bella moves first and colors one uncolored edge white on every move; after each of her moves, Chingiz colors two uncolored edges black. After every edge has been colored, Chingiz wins if $\omega(B) > \omega(W)$, and Bella wins otherwise.

Prove or disprove that Chingiz has a winning strategy for every $n \geq 4$.

G. Dynamics and tournaments

28. Rock, Paper, Scissors on a Graph

Difficulty: a) 2/5; b) 3/5; c) 4/5; d) 4/5 · Extended problem by Mark Pimenov, 2026 · Original part a): HSE “Vysshaya Proba” Olympiad, 2024 Final Round, Grade 11, Problem 11.5

Let G be a finite simple connected graph. Each vertex contains a rock, a pair of scissors, or a sheet of paper. Scissors beat paper, paper beats rock, and rock beats scissors.

Every minute, the object at a vertex is replaced by the object that beats it if such an object is present at one or more neighboring vertices. Otherwise it does not change. All replacements are simultaneous: each of them uses the configuration at the beginning of the minute.

- a) Suppose that G is a tree. Prove that for every initial configuration all changes stop after finitely many minutes.
- b) In the setting of part a), suppose that G has n vertices. Determine the greatest possible number of minutes before all changes stop, where the maximum is taken over all trees on n vertices and all initial configurations.
- c) Suppose that G is not a tree. Prove that there is an initial configuration in which changes continue forever.
- d) Replace the three objects by an odd number $m = 2r + 1 \geq 3$ of objects

$$A_0, A_1, \dots, A_{2r},$$

arranged cyclically; indices are taken modulo m . The object A_i loses to $A_{i+1}, A_{i+2}, \dots, A_{i+r}$.

In one minute, an object A_i at a vertex is replaced by A_{i+1} if at least one neighboring vertex contains an object that defeats A_i ; otherwise A_i does not change. All replacements are again simultaneous. Prove the analogues of parts a) and c) for this rule.

29. Two Leagues

Difficulty: a) 2/5; b) 3/5 · Author: Mark Pimenov, 2026

There are n teams in a single round-robin tournament with no draws: every two teams play one match, and exactly one of them wins it. After the tournament, all results are announced. Two leagues then take turns choosing an unselected team and admitting it. League One moves first, and passing is not allowed.

After all teams have been chosen, call a match **cross-league** if its two teams belong to different leagues.

- a) Suppose that n is even. Prove that League One can guarantee that its teams win at least half of all cross-league matches.
- b) For which even values of n can League One guarantee, for every possible set of tournament results, that its teams win strictly more than half of all cross-league matches?