

Origami

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The great Russian mathematician V.I. Arnold formulated many beautiful problems. In particular, in 1956 he formulated the following problem about a crumpled ruble.

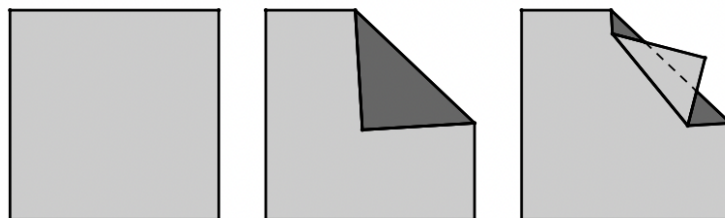
Problem 1. Is it possible to fold a rectangular sheet of paper (a banknote) on a plane so that the perimeter of the resulting figure exceeds the perimeter of the original sheet?



The question immediately arises: what does “*fold*” mean in the formulation? Depending on the different definitions, different answers are obtained.

Exercise. An arbitrary polygon M is bent along a line intersecting it and the result is polygon M' . Prove that its perimeter has not increased.

From the exercise above it immediately follows that it is impossible to increase the perimeter of a given polygon with any number of successive folds. In the figure below, in addition to folds, another operation is shown, which we will call *folding*.

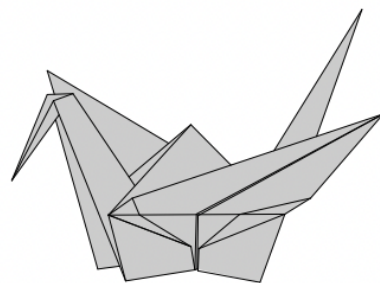
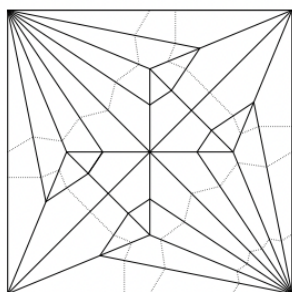


It is obvious that it increases the perimeter, but the following question still remains open

Problem 2. *Is it possible to increase the perimeter of a polygon by successively bending and unbending it?*

Perhaps you will be able to solve it.

By *folding* we can understand an even more general operation: imagine that we have marked the folds in advance and fold the sheet as a whole, so that all the areas into which the sheet is divided by folds remain flat, and the folding occurs only along the folds. The picture below shows the corresponding pattern and a Japanese crane folded from it. The dotted line indicates the folds that are added when straightening the wings, bending the neck, tail and head (the head is the lower right corner, the wings are the lower left and upper right, the tail is the upper left).



In the same way, one can consider any *abstract scheme* of origami (dividing a given rectangle into polygons, where bending is potentially carried out along the sides of the polygons that do not lie on the boundary of the original rectangle). A very non-trivial **question** is *the existence of a realization of this abstract scheme (i.e., is it possible to perform the corresponding bending along the drawn folds)*. It would be interesting to make progress in it. Our goal will be to solve the following two problems.

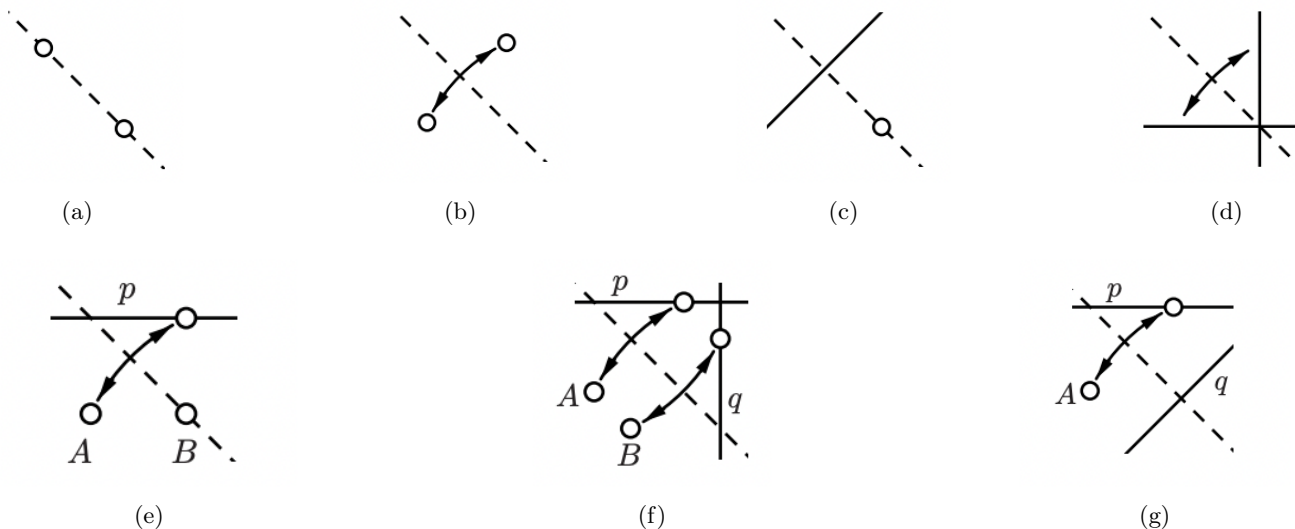
Problem 1. *Prove that there exists an abstract origami scheme that increases the perimeter.*

Problem 2. *Is it possible to make the perimeter arbitrarily large?*

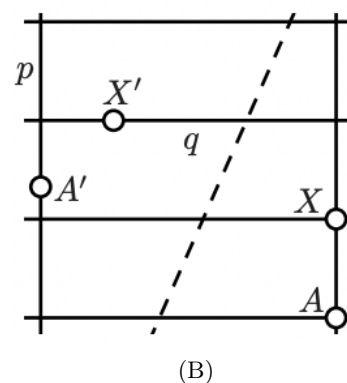
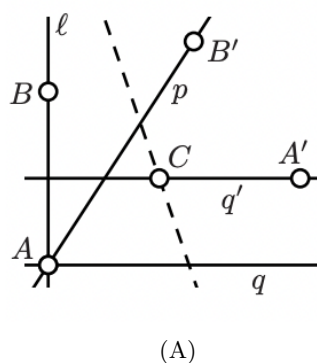
Origami Construction Rules

We will begin by formulating the 7 rules that origami artists use. Like compass and ruler construction, origami construction consists of successive folding of a sheet of paper according to the rules in this list.

1. The sheet can be folded so that the two marked points are on the fold. (Fig. (a)).
2. The sheet can be folded so that one marked point goes into another marked point. (Fig. (b)).
3. The sheet can be folded so that the marked point falls on the fold, and the marked line goes into itself (that is, the fold line will be perpendicular to it). (Fig. (c)).
4. The sheet can be folded so that the marked line passes into another marked line. (Fig. (d)).
5. Let a line p and two points A and B be marked. Then the sheet can be folded so that point B falls on the fold, and A on line p . (Fig. (e)).
6. Let two lines p and q and two points A and B be marked. Then the sheet can be folded so that point A falls on line p , and point B falls on line q . (Fig. (f)).
7. Let two lines p and q and a point A be marked. Then the sheet can be folded so that point A falls on line p , and line q goes into itself (that is, the fold line will be perpendicular to it). (Fig. (g)).



Series A: Origami Constructions



Problem 1. A square sheet of paper with a side of 1 is given. Measure the distance $\frac{5}{6}$ on this sheet.

Problem 2. Three vertices of a triangle are marked on a sheet of paper. Make an origami construction of its circumscribed circle center, orthocenter, and median intersection point and make sure that they lie on one straight line (Euler).

Problem 3. Make an origami construction of a regular triangle.

Problem 4. Fedya marked three points on a sheet of paper. Will Daniil be able to bend the sheet along a certain straight line so that these points are at the vertices of an equilateral triangle?

Problem 5. Make an origami construction of a regular pentagon.

Problem 6. Make an origami construction of an arbitrary regular odd-sided polygon.

Problem 7. Is it possible to make an origami construction of an arbitrary regular even-sided polygon?

Problem 8. (Trisection of an angle) Divide the given angle into three equal parts using the picture (A).

Problem 9. (Doubling the cube) Construct two segments with length ratio $\sqrt[3]{2}$ using picture (B).

Problem 10. Lesha cut out a square sheet of paper with a side of 2024. Will he be able to construct a segment of length 1 by making no more than 20 folds?

Problem 11. Fedya cut out a square sheet of paper with a side of 2016. Will he be able to construct a segment of length 1 by making no more than ten folds?

Problem 12. Construct as many folds as you like that are tangent to some parabola.

Problem 13. Construct as many folds as you like that are tangent to some ellipse.

Problem 14. Construct as many folds as you like that are tangent to some hyperbola.

Problem 15. Prove that for any point that can be constructed using a compass and ruler, it can be constructed by folds.

Problem 16. Given the center of a circle, a point on the circle, and an arbitrary point M different from the center, make an origami construction of the point M' , the inverse of the point M relative to the circle.

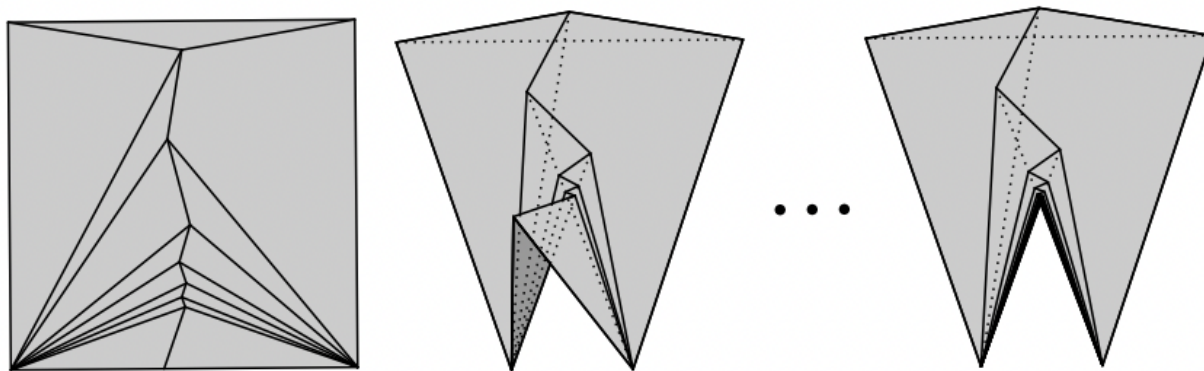
Problem 17. Is it possible to divide this angle into 5 equal parts using origami construction?

Problem 18. Is it possible to divide this angle into 7 equal parts using origami construction?

Problem 19. Is it possible to construct two segments with a length ratio of π ?

Series B: structures that increase the perimeter

The following example is based on Ivan Yashchenko's idea, it also gives only a slight increase in perimeter, but consists of the consistent application of the same simple action. If you repeat it enough times, you will get a figure as close as you like to the figure on the right, its perimeter is slightly larger than that of a square.



Problem 1. Understand this construction and prove that this process does indeed increase the perimeter. Estimate the number of repetitions after which the perimeter becomes greater than the perimeter of the original square. Find the number of layers in the thickest place of the resulting figure.

Let's assume that a square is folded into a flat figure. Let's mark all its folds and unfold it. The folds give a partition of the square into polygons, as, for example, in the left picture in the figure above. This partition is called a *fold grid*.

Problem 2. Prove that:

- the polygons in the fold grid can be colored black and white so that adjacent polygons on the side have different colors;
- in this case, the sum of the black angles converging at one internal vertex of the grid is equal to 180° ;

c) construct a partition of the square into polygons such that conditions 1 and 2 are satisfied, but at the same time it is not a fold grid.

Problem 3. Suppose a convex polygon M can be folded on the plane to form another convex polygon M_0 . Prove that the perimeter of M_0 does not exceed the perimeter of M . (In particular, a square cannot be folded to form a convex polygon with a larger perimeter.)

Let us call a figure F an insert if it satisfies the following condition: any plane figure that can be folded from F can be completely covered by a copy of F . In other words, if a figure F' is folded from F , then F' is congruent to a subset of F .

Problem 4. Is it true that circles and only they are inserts?

Series B: flexible and rigid polyhedrons, their volumes

Heron's formula is widely known, expressing the area S of a triangle through the lengths of its sides a, b, c :

$$S = \sqrt{p(p-a)(p-b)(p-c)}$$

It can be rewritten as:

$$S^2 = \frac{1}{16}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)$$

Unfortunately, this formula does not apply to arbitrary polygons. However, one can try to consider polygons of some special type, for example, inscribed ones.

Problem 1. Derive a similar formula expressing the area through the lengths of the sides for an inscribed quadrilateral.

Problem 2. Is there a similar formula expressing the area in terms of the lengths of the sides for an inscribed polygon?

And what happens in space?

Problem 3. Derive the following formula expressing the volume of a tetrahedron through the lengths of its edges:

$$\begin{aligned} V^2 = \frac{1}{144} [& l_1^2 l_5^2 (l_2^2 + l_3^2 + l_4^2 + l_6^2 - l_1^2 - l_5^2) + \\ & + l_2^2 l_6^2 (l_1^2 + l_3^2 + l_4^2 + l_5^2 - l_2^2 - l_6^2) + l_3^2 l_4^2 (l_1^2 + l_2^2 + l_5^2 + l_6^2 - l_3^2 - l_4^2) - \\ & - l_1^2 l_2^2 l_4^2 - l_2^2 l_3^2 l_5^2 - l_1^2 l_3^2 l_6^2 - l_4^2 l_5^2 l_6^2] \end{aligned}$$

Let us call *bending of a polygon* a continuous deformation on a plane in which at least one of the angles changes, but the edges remain equal to the original ones.

It is easy to understand that a triangle is a *rigid figure*, i.e. it does not allow bending. At the same time, an arbitrary polygon with a large number of vertices is already bendable. And what happens in space? Let us recall some definitions.

A *polyhedron* is a set of a finite number of flat polygons in three-dimensional Euclidean space such that:

a) each side of any of them is a side of exactly one more polygon;

- b)** from any of the polygons one can move to any other along a chain of polygons in which successive polygons have a common side;
- c)** if two polygons have a common vertex, then the chain connecting them can be composed of polygons that all have this vertex.

Note that a polyhedron can have *self-intersections*, i.e. the polygons that form it can intersect.

Let us call *bending of a polyhedron* a continuous deformation in space in which at least one of the dihedral angles at the edges changes, but the faces remain equal to the original ones.

Problem 4. Let $ABCD$ be a quadrilateral in space with equal opposite sides $AB = CD$, $AD = BC$, whose vertices do not lie in the same plane. Let l denote the line passing through the midpoints of the diagonals AC and BD . Let N be an arbitrary point in space different from A, B, C, D and not lying on the l axis. Let S be a point symmetric to N with respect to l . Connect S and N with segments with the vertices of the quadrilateral $ABCD$. Consider the resulting octahedron $ABCDSN$. Prove that it admits bending.

Problem 5. Let a quadrilateral $ABCD$ be given in space with pairwise equal sides $AB = AD$, $CB = CD$. Consider a plane p passing through the diagonal AC perpendicular to the diagonal BD . Take any point N not lying on the plane p and different from the vertices of the quadrilateral $ABCD$. Let S be a point symmetrical to N with respect to the plane p . Connect the points S and N with segments with the vertices of the quadrilateral $ABCD$. Consider the resulting octahedron $ABCDSN$. Prove that it admits bending.

Note that the octahedra from the previous problems have self-intersections. There is exactly one more type of flexible octahedron (also with self-intersections). These octahedra were discovered by the Belgian engineer René Bricard in 1897. Are there flexible polyhedra without self-intersections?

In 1975, American mathematician Robert Connelly figured out how to get rid of the intersection (the so-called "Connelly notches"), and "true" flexible polyhedra appeared. In 1978, German mathematician Klaus Steffen found a polyhedron without self-intersections, with only 9 vertices and 14 faces.

Problem 6. Glue the Steffen polyhedron according to the pattern below and make sure it allows for bending.

