

Poncelet Invariants

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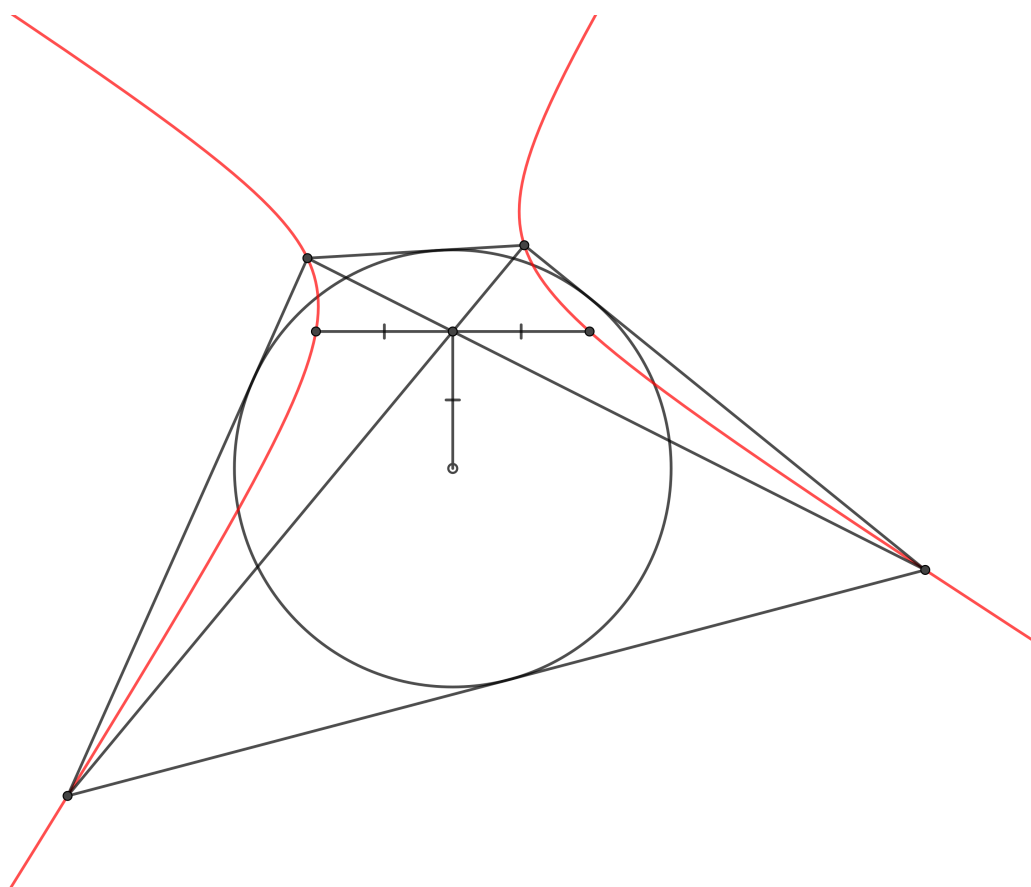
Cool ratio lemma

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Introduction

The Poncelet theorem is one of facts of projective algebraic geometry. This famous theorem is an example of closing theorems, when some process of constructing of new points by drawing several lines is periodic. The theorem has many different proofs, variants, and generalizations. In spite of its notoriety, this theorem presents a very difficult fact, which can be proved by many different substantial ways. The most known synthetic proof using the pencils of circles can be found in [1], and the analytic proof using the density function in [8].

All scholars advanced in geometry hear about the Poncelet theorem at least one time. Some of them know several proofs of the theorem, and the most advanced know the different invariants appearing in Poncelet rotation. For example the common point of diagonals of the Poncelet quadrilateral is fixed... The construction of the theorem is very deep, and allows to found new results, some of them are very complicated and substantial. One of such results was found and proved by the authors of the project, and we will lead to it through the subjects which seem not connecting with the project. Note also that many proofs of the Poncelet theorem have one defect — they use euclidean notions of segments and angles. In the project we will use euclidean, projective, and algebraic reasoning, and consider the Poncelet invariants on the different hands.

Such euclidean instrument as cool ratio lemma (CRL) doesn't seem connecting with the Poncelet theorem. This lemma was formulated for the first time in the paper [11], published in 2020 on the known site aops.com. At first sight this is a simple calculating instrument. But the detail considering allows to discover some projective properties of CRL and its liaison with the addition of points on a cubic curve. It is shown that CRL remember the cross-ratios and has a deep nature connecting with the notion of *divisor* of the cubic, or, more exactly, its analog for the degenerated cubic. We will not submerge in this theory, but the advanced readers can think about the description of CRL in the terms of the analog of the divisor.

Also note that in the first five paragraphs of the project we will not work with non-degenerated cubics, hence we don't suppose any knowledge in this region from the participants.

But the main — in process of thinking about one problem formulated 11 years ago and solved recently by the authors, a complicated, nice theorem connecting connecting CRL and the Poncelet theorem was found.

In the project we will understand, what is CRL and how use it , prove the Poncelet theorem, and discuss their mysterious liaison. Also we will make a deep submerge in the world of euclidean Poncelet:

- formulate and prove new or few known invariants;
- educate the connection of the Poncelet theorem with the Clawson conjugation;
- find the (probably) new proof of the known Aiyer theorem;
- discuss the using of the Poncelet theorem in the problems which seem not connecting with it;

- present the new method of solving some problems using the latent Poncelet and basing on the rotation of the triangle around some conic, not given in the problem. This method allows to solve many difficult olympiad problems (for example 10.4 of Sharygin olympiad, 2025);
- and many other...

This is not the first appearing of the Poncelet theorem in the Summer conference. The project was dedicated to it in 2014 [4]. Our project is completely independent, hence the knowledge of the previous project is not necessary. But the most motivated participants can see the 2014 year project for the deep submerge into the atmosphere of the Poncelet theorem.

The structure of the project is the following:

The parts 1, 2, and 3 are given at the beginning of the Conference, and the parts 4, 5 after the intermediate finish.

The part 1 contains all necessary theoretic base. Since the project contains several different parts, the materials of paragraph 1 are also different by the themes and the difficulty. This has not to fright you. You can miss this paragraph and return to it when it is necessary.

In the part 2 we discuss in detail the cool ratio lemma and its applying in the olympiad problems, and in the part 4 we consider the projective nature of CRL and its liaison with the addition of points on the degenerated cubic. These two paragraphs are not connected with the Poncelet theorem, hence you can solve them, even if you are not interested to think about the remaining parts of the project.

The part 3 is dedicated to different invariants of the Poncelet theorem and its applying in the olympiad problems.

In the part 5 we demonstrate the liaison between the cool ratio lemma and the Poncelet theorem. We formulate and prove a complicated and very nice result at the joint of these two worlds, and consider several examples of its using.

Finally, in the additional paragraph we present three few known proofs of the Poncelet theorem. Two of them use the addition of points on the cubic and/or the exit into the space. The third, more algebraic proof of the theorem and some connected facts will be presented as the sequence of problems for the most advanced readers.

1 Necessary information

This paragraph contains necessary definitions of notions which will be used and important facts about these notions. We concentrate on the assertions and the constructions of the projective geometry, because the projective language will be actively used later. The assertions of these paragraph may be used without proof, but you may discuss with the tutor the facts which are not known to you.

1.1 Projective geometry

There we cite necessary facts of projective geometry. We restrict by the dimensions 1 and 2.

Definition 1.1. Let O be an arbitrary point of the plane. *The pencil of lines $\mathcal{L}_O$ centered at O is the set of all euclidean lines passing through O .*

Consider the pencil $\mathcal{L}_O$ and an arbitrary line ℓ , not passing through O . Then all lines of pencil $\mathcal{L}_O$, except one, intersect ℓ . Thus we obtain the correspondence between the points of ℓ and the lines of $\mathcal{L}_O$. This motivates the following

Definition 1.2. *The projective line is an arbitrary pencil of lines $\mathcal{L}_O$. Any line of this pencil is called a projective point. If ℓ is an arbitrary euclidean line, not passing through O , then the projective point $J \in \mathcal{L}_O$, corresponding to the line parallel to ℓ , is called the infinite point of ℓ . Allowing some liberty we will say that the euclidean line ℓ is completed by the infinite point J .*

Now define the cross-ratio of four projective points on a projective line. Let ℓ_1, ℓ_2, ℓ_3 , and ℓ_4 be four projective points corresponding to four euclidean lines of the pencil $\mathcal{L}_O$.

Definition 1.3. *The cross-ratio of the quadruple of projective points ℓ_1, ℓ_2, ℓ_3 и ℓ_4 is the number*

$$[\ell_1, \ell_2; \ell_3, \ell_4] = \frac{\sin \angle(\ell_1, \ell_3)}{\sin \angle(\ell_2, \ell_3)} : \frac{\sin \angle(\ell_1, \ell_4)}{\sin \angle(\ell_2, \ell_4)}.$$

If the pencil $\mathcal{L}_O$ is intersected by a line ℓ , then the lines ℓ_1, ℓ_2, ℓ_3 , and ℓ_4 strike on ℓ points A, B, C , and D respectively. We obtain that $[\ell_1, \ell_2; \ell_3, \ell_4] = \frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$. This allows to define the cross-ratio on the euclidean line.

Definition 1.4. *The cross-ratio of four collinear points A, B, C, D is the number $[A, B; C, D] = \frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$. The quadruple A, B, C, D is called a harmonic quadruple, if $[A, B; C, D] = -1$.*

It is convenient to consider a projective line as a usual euclidean line completed by the infinite point J . Then for example a quadruple A, B, M, J is harmonic, if M is the midpoint of segment AB .

The following fact is an important property of cross-ratio: if $[A, B; C, D_1] = [A, B; C, D_2]$, then $D_1 = D_2$. Thus the cross-ratio is the analog of the euclidean distance, but this projective «distance» is measured with respect to three points.

Definition 1.5. Let ℓ_1 and ℓ_2 be projective lines. A map $f: \ell_1 \rightarrow \ell_2$ is called *projective*, if it conserves cross-ratios, i.e. for any quadruple A, B, C, D we have $[A, B; C, D] = [f(A), f(B); f(C), f(D)]$.

Theorem 1.1. Any projective map of a line to itself is uniquely defined by the images of three points. I.e., for any points $A, B, C, A', B',$ and C' there exists a unique projective map f , such that $f(A) = A', f(B) = B'$ and $f(C) = C'$.

Now remember the notion of the projective plane. Consider a pencil of lines in the space, i.e. the set of lines passing through the fixed point O . Such set of lines is called *the projective plane*. Each line passing through the fixed point is called *a projective point*, and each plane passing through this point is called *a projective line*. Note that this definition is concordant with the definition given above of the projective line as a pencil of lines on the plane. If we take an usual plane α not passing through the center O of the pencil, then each projective point and projective line strike an usual point and a usual line respectively on it. All projective points not intersecting α are called *infinite* and lie on *the infinite projective line* α_∞ .

Sometimes the analytic view to the projective points and lines can be useful. Fix on the space a coordinates system $Oxyz$ with the origin at the center O of our pencil. Compare each projective point with the direction vector of the corresponding euclidean line. Since this vector is defined with exactness to a non-zero coefficient, the coordinates of a projective point are also defined with exactness to multiplying into a non-zero constant. Such coordinates are called *homogeneous* and are denoted by $(x : y : z)$. If the plane α is defined in the coordinates system $Oxyz$ by the equation $z = 1$, then the projective point $(x : y : z)$ strikes on it the point with euclidean coordinates (x, y) , and the infinite projective line α_∞ is defined by the equation $z = 0$.

Similarly each projective line is defined by an equation $ax + by + cz = 0$ and can be given by the numbers $a, b,$ and c . But this triplet of numbers (Which are the coordinates of the normal vector of this plane) is also defined with exactness to a non-zero coefficient. Hence our plane (and therefore projective line) corresponds to the triplet of homogeneous coordinates $(a : b : c)$. The point $(x : y : z)$ lies on the plane $(a : b : c)$ if and only if $ax + by + cz = 0$.

Note the equivalency of coordinate descriptions of projective points and lines. This equivalency can be formalized by the following assertion (*the projective duality*).

Theorem 1.2 (The projective duality). If some assertion of the projective geometry is correct, then the dual assertion obtained by replacing of any word «point» to «line» and any word «line» to «point» is also correct.

Remember now the notion of projective map of the projective plane.

Definition 1.6. A *projective map of the projective plane* is a map transforming all projective lines to projective lines.

Projective maps can be defined in the homogeneous coordinates by the following way. Consider a *linear map* of the space, i.e. a map transforming the coordinates of each point linearly:

$$(x, y, z) \mapsto (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z),$$

where a_{ij} are some real numbers. This map generates the projective map transforming the homogeneous coordinates by the following way:

$$(x : y : z) \mapsto (a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z).$$

The euclidean coordinates on the plane α are transformed, similarly to the case of the projective line, by fractionally linear formulas:

$$(x, y) \mapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} \right).$$

There exists also the theorem allowing to define a projective map of the projective plane by the images of several points.

Theorem 1.3. For any two quadruples of points in general position (A, B, C, D) and (A', B', C', D') there exists a unique projective map transforming the first quadruple to the second one.

1.2 Conics

We remember in this paragraph the main notions connected with *the conics*. It is important that our main object is the projective plane (although the conics on the euclidean plane can also be considered in the problems), hence we need to give the definition of conics connected with the projective geometry, on the other hand it has to be formulated without previous preparation. Such definition is the following.

Definition 1.7. A *conic $\mathcal{C}$ on the plane* is the set of zeros of a polynomial $F \in \mathbb{R}[x, y]$ with degree 2 on variables x and y .

Depending on the polynomial F we can obtain *degenerated conics* (corresponding to reducible polynomials and presented by pairs of lines) and *non-degenerated conics* (corresponding to irreducible polynomials, which can be ellipses, hyperbolas, or parabolas). Later, if the converse is not indicated, we consider only non-degenerated conics.

Conics can be considered not only on the usual plane, but on the projective one. On the geometrical point of view this means, that we have to exit into the plane and to pass the line through a fixed point

and all points of the given conic. As a result we obtain *the projective conic* $\widehat{\mathcal{C}}$, which is in fact the cone generated by the given conic. It is not convenient to work with such object (Can you inscribe a triangle into such conic?..). Hence the analytic language, not demanding a geometric description will be useful here. Since an equation $F(x, y) = 0$ on the usual plane give some condition for the coordinates (x, y) , we have to obtain the condition for the homogeneous coordinates $(x : y : z)$ on the projective plane. We obtain this by the following way. Let the conic $\mathcal{C}$ is defined by the polynomial $F(x, y)$, i.e. $\mathcal{C} = \{(x, y) : F(x, y) = 0\}$. Replace $F(x, y)$ to the homogeneous polynomial $\widehat{F}(x, y, z) = z^2 F(x/z, y/z)$ on three variables. The set of zeros of $\widehat{F}(x, y, z)$ is the projective conic $\widehat{\mathcal{C}}$: $\widehat{\mathcal{C}} = \{(x, y, z) \neq (0, 0, 0) : \widehat{F}(x, y, z) = 0\}$.

Later we will use the same designation for euclidean and corresponding projective conics. It will be clear from the context, do we consider the conic in the euclidean or the projective geometry.

Not also the following assertion:

Statement 1.8. A projective map of a circle is a conic, conversely, an arbitrary conic can be obtained as a projective map of a circle.

The first part is clear because a projective map do not increase the degree of the polynomial. The second one can also be proved by algebraic methods.

Sometimes the disposition of the conic with respect to the infinite line ℓ_∞ will be important.

Definition 1.9. An *ellipse* not intersecting the infinite line. A *hyperbola* is a conic meeting the infinite line at two different points. Finally, a *parabola* touching the infinite line.

Theorem 1.4. Any five points in general position define a unique conic passing through them. Any 5 lines in general position define a unique conic touching them.

The notion of cross-ratio also can be defined on the conic.

Definition 1.10. Let A, B, C, D be four points on the conic $\mathcal{C}$. The *cross-ratio* $[A, B; C, D]$ is the cross-ratio of lines $[XA, XB; XC, XD]$, where X is an arbitrary point of the conic, distinct from A, B, C , and D .

The correctness of this definition can be proved by the following way. Transform the conic $\mathcal{C}$ into a circle by some projective map. This map conserve the cross-ratio, hence it is sufficient to prove the correctness for circles. And this is simple (we can use the equalities of inscribed angles or, applying an inversion, reduce the problem to the cross-ratio of collinear points).

This yields the following important assertion.

Theorem 1.5 (The Sollertinsky lemma). Let $\mathcal{L}_O$ be a pencil of lines, and f be the projective map of the plane. Denote by $\mathcal{L}_{O'}$ the image of $\mathcal{L}_O$ about f . Consider for each line $\ell \in \mathcal{L}_O$ the common point of ℓ and $f(\ell)$. Then the locus of these points is a conic passing through O and O' .

Remark. If $f(OO') = OO'$, the conic is the union of the line OO' and some other line.

Since the cross-ratio of four points is correctly defined on the conic, all facts about cross-ratios formulated above for lines are also correct for conics. For example a projective map of a conic to itself is uniquely defined by the images of three points. Really a conic is isomorphic to a line as a projective manifold.

Consider now the notions of pole and polar with respect to the conic. Again we will use an analytic definition. (This definition can be unusual for you, hence you can consider the poles and the polars applying the standard approach: using complete quadrilaterals, tangents and harmonic quadruples. But the analytical definition has several advances, thus it is useful to know it).

Definition 1.11. Let $\mathcal{C}$ be the conic defined by the equation $q(v) = 0$, and u be an arbitrary vector. Consider the bilinear form $Q(u, v) = \frac{q(u+v) - q(u) - q(v)}{2}$ (in fact this strange formula contains the formula of square of sum). Two vectors u and u' are called *conjugated with respect to $\mathcal{C}$* , if $Q(u, u') = 0$. For any fixed vector u the set of all conjugated vectors u' is called *the polar* of the point corresponding to u .

Note that the equation of the polar is $Q(u, v) = 0$, where Q and u are fixed. Thus it is linear on v , i.e. the polar is some projective line. Conversely, let ℓ be an arbitrary projective line. Choose two non-collinear vectors v_1 and v_2 on it (i.e. two different projective points). Then there exists a unique (with exactness to the non-zero coefficient) vector u , such that $Q(u, v_1) = Q(u, v_2) = 0$: vector u corresponds to the common point of the polars of v_1 and v_2 . This point is called *the pole of line ℓ* .

Thus the conic $\mathcal{C}$ defines the correspondence between the points and the lines: each point corresponds to its polar, and each line corresponds to its pole. Such correspondence is called *polar*.

The image of the infinite line ℓ_∞ about the polar correspondence with respect $\mathcal{C}$, is the point called *the center of the conic $\mathcal{C}$* .

Note the following classical fact.

Theorem 1.6 (The duality principle). Let $\mathcal{C}$ be an arbitrary conic, A and B be arbitrary points, a and b be their polars. Then A lies on b if and only if B lies on a .

This assertion clearly follows from the symmetry of Q : $Q(u, v) = Q(v, u)$.

About the complex word. All notions considered above can be defined similarly for the complex numbers (really for an arbitrary field). We can define the complex projective line $\mathbb{CP}^1$, the complex projective plane $\mathbb{CP}^2$, the notions of conics, polars, projective maps, etc; the question is, how define the cross-ratio. We will not give the details of this theme — you can find them in [3]. We have to understand the cross-ratio as a function on four variables, which change similarly to the real case when the variables are rearranged, and is conserved by the projective maps and polar transformations. Also $\mathbb{CP}^1$ can be considered as a sphere — in fact we completed the plane by one point, thus the complex projective line can be obtained as the stereographic projection of the sphere to the plane. It is convenient to consider the complex projective plane because any 2 conics on it have 4 common points, any conic has 2 points

with any line, and any circle passes through 2 fixed points with homogeneous coordinates $\mathcal{J}_+(1 : i : 0)$ and $\mathcal{J}_-(1 : -i : 0)$, they are called *circular points*. And we can present this as usually. You can see the details in [2].

The following theorem allows to reduce some assertion about conics to the case of two circles:

Theorem 1.7.

- (a) Any two conics on $\mathbb{CP}^2$ may be transformed into two circles by some projective map.
- (b) If two conics on $\mathbb{RP}^2$ have at most two common point (calculating their divisibility), they can be transformed into two circles by some real projective map.

1.3 Euclidean opinion about the conics

Sometimes it is useful to consider the conics from euclidean point of view. Discuss briefly this opinion. Detailed information can be found in [1] (chapters 1 and 3.2).

Definition 1.12. Let points P, Q and a real number $k > |PQ|$ be given. *Ellipse* with foci P and Q is the locus of points A , such that $AP + AQ = k$.

Definition 1.13. Let points P, Q and a real number $k \in (0, |PQ|)$ be given. *Hyperbola* with foci P and Q is the locus of points A , such that $|AP - AQ| = k$.

Definition 1.14. Let a point F and a line ℓ be given. *Parabola* with focus F and directrix ℓ is the locus of points A , such that $AF = \text{dist}(A, \ell)$.

In some sense we can think, that the parabola has second infinite focus. It is not difficult to prove that these definitions are equivalent to the given in subsection (1.2) ones. We will not to consider the details.

The following simple properties are very important. Tare an attention to the case of the parabola — our non-formal reasoning about its second infinite focus obtains there a new development.

Statement 1.15 (Optical property of conics).

- (a) Let a point A lie on the conic with foci F_1 and F_2 . Then the tangent to the conic at A is the bisector (external for the ellipse, internal for the hyperbola) of angle F_1AF_2 .
- (b) Let a point A lie on the parabola with focus F and directrix ℓ . The line m passes through A and is perpendicular to the directrix. Then the tangent to the parabola at A bisects the angle between AF and m .

Statement 1.16 (Isogonal property of conics).

- (a) Let a point A lie outside the conic with foci F_1 and F_2 . Then the tangents from A to the conic are isogonals with respect to the angle F_1AF_2 .

- (b) Let a point A lie outside the parabola with focus F and directrix ℓ . The line m passes through A and is perpendicular to the directrix. Then the tangents from A to the parabola are isogonals with respect to the angle between AF and m .

Corollary 1.17.

- (a) Let the conic with foci F_1 and F_2 touche the sidelines of the triangle ABC . Then F_1 and F_2 are isogonally conjugated with respect to the triangle ABC .
- (b) Let the parabola with focus F and directrix ℓ touche the sidelines of the triangle ABC . Then F lies on the circumcircle of the triangle ABC , and ℓ passes through its orthocenter.

Discuss the liaison of the conics and the polar correspondence.

Theorem 1.8. Let a point O and a circle γ be given. Then the image of γ with respect to the polar correspondence about an arbitrary circle centered at O is a conic with one of foci coinciding with O . If O lies inside γ , this conic an ellipse, If O lies outside γ it is a hyperbola, and if O lies on γ , it is a parabola. Conversely: if F is the focus of a conic Γ , then the image of Γ with respect to a polar correspondence centered at F is a circle.

1.4 Isogonal conjugation in the quadrilateral

Definition 1.18. The points P and Q are called *isogonally conjugated with respect to the polygon* $A_1A_2 \dots A_n$, if the directions from each vertex of the polygon to these points are isogonal with respect to the corresponding angle.

Theorem 1.9. Let a quadrilateral (or, generally, closed quadrilateral broken line) $ABCD$ and a point P be given. Then the following assertion are equivalent:

- (a) There exists a point Q , isogonally conjugated to P with respect to $ABCD$.
- (b) $\angle APB = \angle DPC$
- (c) The projections of P to the sidelines of the quadrilateral are concyclic or collinear.
- (d) There exists a conic with focus P , touching the sidelines of the quadrilateral.

Remark. Take an attention, in (c) the projections can be not only concyclic, but collinear. There exists exactly one point, such that its projections are collinear — the Miquel point. Formally this is not a contradiction: using the simple count of angles we can see, that there exists a point isogonally conjugated to the Miquel point, this point is infinite. How find the direction to this point? We obtain the answer from the following theorem:

Theorem 1.10. The Miquel point of the quadrilateral is isogonally conjugated to the infinite point of its Gauss line.

You can read the proof of the theorem (1.9) and to make a detailed acquaintance with the isogonal conjugation in the quadrilateral in [9].

1.5 Projective involutions

Definition 1.19. *The involution* is a map $f : M \rightarrow M$ from an arbitrary set M to itself, such that $f \circ f = \text{id}$. I.e, $f(f(x)) = x$ for any $x \in M$. If the cross-ratio is defined on M (for example, if M is a projective line, a pencil of lines, or a conic), then it is possible that f conserves the cross-ratios (i.e. for any different $a, b, c, d \in M$ we have $[a, b; c, d] = [\varphi(a), \varphi(b); \varphi(c), \varphi(d)]$). In this case f is called *the projective involution*.

Statement 1.20. Two projective involutions coinciding on two pairs of elements are equal.

Proof. Let φ and ψ be projective involutions, $\varphi(a) = \psi(a) = b$, $\varphi(c) = \psi(c) = d$, $\varphi(x) = y$, $\psi(x) = z$. Then $[a, b; c, y] \stackrel{\varphi}{=} [b, a; d, x] \stackrel{\psi}{=} [a, b; c, z]$, which yields that y and z coincide. $\square$

Statement 1.21.

- (a) Any projective involution of the projective line is an inversion or a central symmetry.
- (b) Any projective involution of the circle is a central projection.

Proof. It is clear that all mentioned maps are projective involutions, now the assertion follows from (1.20). $\square$

1.6 The Clawson conjugation

We begin this subsection by a new definition of the Miquel point. It is most universal than the standard one because it works for degenerated quadrilaterals

Definition 1.22. Let points A, B, C, D be given on the plane. Some of them can be collinear, also the points in one of pairs (A, C) , (B, D) can coincide. *The Miquel point of the quadruple* (A, B, C, D) is the center of the spiral similarity, mapping the segment AB to the segment DC , or, which is equivalent, the center of the spiral similarity, mapping the segment AD to the segment BC . Sometimes, allowing some liberty we will call the Miquel point of the quadruple (A, B, C, D) «the Miquel point of the quadrilateral $ABCD$ ».

Remark. If A, B, C, D are different points in general position, this definition coincides with the standard one.

Remark. Remember that the B-Dumpty point of the triangle ABC is the projection of its circumcenter to the symmedian from B . It is easy to prove that if S is the B-Dumpty point, then $\angle SAB = \angle SBC$ and $\angle SCB = \angle SBA$. Therefore the B-Dumpty point is the Miquel point of «the quadrilateral» $ABCB$.

Let A, B, C, D be points in general position on the plane. The lines AB and CD meet at point E , the lines BC and AD meet at point F , the lines AC and BD meet at point G . Let M be the Miquel point of (A, B, C, D) . All talked above yields that $MA \cdot MC = MB \cdot MD = ME \cdot MF$, and the bisectors of angles $\angle AMC$, $\angle BMD$, and $\angle EMF$ coincide. Therefore there exists an inversymmetry centered at M , which swaps A and C , B and D , E and F . This map is called *the Clawson conjugation* with respect to $ABCD$.

Remark. If $BC \parallel AD$, then M coincides with E , and the Clawson conjugation is the inversymmetry about the trapezoid $ABCD$.

Statement 1.23. Let points P and Q be conjugated by Clawson with respect to $ABCD$. Then $\angle APB = \angle ADQ + \angle QCB$.

Proof. Since the Clawson conjugation maps A to C and P to Q , the triangles MAP and MQC are similar. Also the triangles MBP and MQD are similar. Therefore $\angle APB = \angle APM + \angle MPB = \angle QCM + \angle MDQ = (\angle QCB + \angle BCM) + (\angle MDA + \angle ADQ) = \angle ADQ + \angle QCB$. The last equality follows from the similarity of triangles MBC and MAD . $\square$

The assertion (1.23) can be used for the control of the Clawson conjugation. For this it is sufficient to examine that the equality is correct for two adjacent sides. Note now an important corollary:

Corollary 1.24. The Clawson conjugation swaps the points isogonally conjugated with respect to the quadrilateral.

Proof. Let P and Q be isogonally conjugated with respect to $ABCD$. Then $\angle APB = \angle DPC = \angle PDC + \angle DCP = \angle ADQ + \angle QCB$. Similar equalities are correct for the remaining sides, therefore P and Q are conjugated by Clawson. $\square$

Statement 1.25. The Clawson conjugation is a projective involution on $\mathbb{CP}^1$.

Proof. Clearly it is an involution, prove the projectivity. Let m be the complex coordinate of the Miquel point. If the conjugation maps $P(p)$ to $Q(q)$, then $(m - a)(m - c) = (m - p)(m - q)$. Therefore the map $p \mapsto q$ is fractionally-linear, thus it conserves the cross-ratios. $\square$

The previous assertion and the assertion (1.20) yield, that the Clawson conjugation is *a unique* involution on $\mathbb{CP}^1$, swapping the opposite vertices of the quadrilateral.

Now let $ABCD$ be a cyclic quadrilateral with circumcenter O . From the Brocard theorem we have that O is the orthocenter of triangle EFG , and M be the projection of O to EF . Therefore MO coincides with the symmetry axis, and the Clawson conjugation maps G to O .

To make more detailed education of the Clawson conjugation see [6]

1.7 Classical proof of the Poncelet theorem

Here and later we denote by $\text{pow}_\omega(X)$ the power of the point X with respect to the circle ω .

Theorem 1.11 (Coaxial circles lemma). Let circles ω_1 and ω_2 and a real number k be given. Then the locus of points P , such that $\text{pow}_{\omega_1}(P) = k \cdot \text{pow}_{\omega_2}(P)$ is a circle (a line for $k = 1$), coaxial with ω_1 and ω_2 .

Theorem 1.12. Let $A_1A_2A_3 \dots$ be a Poncelet trajectory, i.e. points of the circle (or the conic) Ω such that the rays $A_1A_2, A_2A_3 \dots$ touche the circle (or the conic) ω . Let $B_1B_2B_3 \dots$ be another Poncelet trajectory. Then the lines $A_1B_1, A_2B_2 \dots$ touche some circle (or conic) coaxial with Ω and ω .

The classical synthetic proof of the Poncelet theorem is based on the theorem (1.12), which follows from the coaxial circles lemma. You can read it for example in citeZasl.

Note also two simple, but important corollaries of the theorem (1.12):

Theorem 1.13. Let $A_1A_2 \dots A_n$ be a polygon inscribed into a circle (or a conic) Ω and circumscribed around a circle (or a conic) ω . Then for any natural $k < n$ all lines A_iA_{i+n} touch some circle (or conic), coaxial with Ω and ω .

Remark. Of course the assumption of closing can be omitted, and the general assertion is correct: if the broken line $A_1A_2 \dots A_k$ rotates between two circles (or conics) Ω and ω , then the line A_1A_k touches for any time some circle (or conic), coaxial with Ω and ω .

Theorem 1.14. Let $A_1A_2 \dots A_{2n}$ be a polygon inscribed into a circle (or a conic) Ω and circumscribed around a circle (or a conic) ω .

- (a) The main diagonals A_iA_{i+n} concur at the limit point of the pencil containing Ω and ω .
- (b) Let A_iA_{i+1} touche the inconic at point X_i . Then the lines X_iX_{i+n} concur at the same limit point.

Our project propose another, more algebraic approach for the proof of the Poncelet theorem, the theorem (1.13), and the theorem (1.14), it is based on $2 - 2$ -correspondences. We give its detailed description in the additional section.

2 Cool ratio lemma

Before solving the problems make acquaintance with the designations. The problems marked by one asterisk have (in the subjective opinion of the authors) relatively high difficulty. If the problem contains several points, some of them can be marked by the asterisk. The solutions of problems marked by two asterisks are unknown to the authors. May the force be with you!

2.1 Definitions and first CRL

We begin by a classical assertion known as *the Odnakov lemma*, *the sines ratio lemma*, or *boring ratio lemma*.

Theorem 2.1 (Boring ratio lemma). Let a triangle AXY and a point C on the sideline XY be given. Then $\frac{XC}{YC} = \frac{AX \cdot \sin(\angle XAC)}{AY \cdot \sin(\angle YAC)}$

Consider now the sines ratio in the right part. Let AC meet for the second time the circumcircle of (AXY) at point B . By the sines law $\frac{\sin(\angle XAC)}{\sin(\angle YAC)} = \frac{BX}{BY}$, which yields the following theorem.

Theorem 2.2 (Cool ratio lemma). Let a circle and its chord XY be given. Points A and B lie on the circle. The lines AB and XY meet at point C . Then $\frac{XA}{AY} \cdot \frac{XB}{BY} = \frac{XC}{CY}$

This formulating has one defect, the lemma can not be used conversely: if both points C and D lie on the circle or on the line XY , then the equality of ratios $\frac{XC}{YC}$ and $\frac{XD}{YD}$ does not yield that C and D coincide. Let us correct this.

Определение 2.1. Fix points X and Y . The line XY divides the plane into two semiplanes. Call one of them *the upper*, and the remaining one *the lower*. *Cool-ratio function about the base XY* is the function $f_{XY}(T)$ from $\mathbb{R}^2 \setminus Y$ to $\mathbb{R}$, with an absolute value $\frac{XT}{YT}$, having a sign $-$, if T lies on the segment XY or the lower semiplane, and a sign $+$ otherwise.

Remark. Take an attention, that f_{XY} and f_{YX} are **not** the same!

Remark. In the problems we will often define the cool-ratio-function for a unique segment. To simplify the exposition we will omit in this case the index XY , announcing previously which segment is used as a base.

Statement 2.2. Let ω be a circle or a line passing through X and Y . Then f_{XY} is a bijective map from $\omega \setminus Y$ to $\mathbb{R}$.

Now, considering different cases, we can formulate the following theorem.

Theorem 2.3 (Cool ratio lemma). Let a circle ω and its chord XY be given. Points A and B lie on the circle, and a point C lies on the line XY . Then A, B, C are collinear if and only if $f_{XY}(A) \cdot f_{XY}(B) = f_{XY}(C)$.

But it is not necessary to consider the cases.

Proof. Let T be the common point of the tangent at A to ω and XY . The collinearity of A , B , and C is equivalent to the equality $[X, Y; A, B] = [X, Y; T, C] \iff \frac{f(A)}{f(B)} = \frac{f(T)}{f(C)} \iff \frac{f(A)}{f(B)} = \frac{f^2(A)}{f(C)} \iff f(A) \cdot f(B) = f(C)$. $\square$

This result allows to prove very non-trivial facts.

Example (The butterfly theorem). Let points A, B, C, D lie on a circle ω . Let P be the common point of AC and BD . Let XY be a chord of ω such that P lies on XY , and $PX = PY$. Let XY meet AD and BC at points M and N respectively. Prove that $PM = PN$.

Proof. Define the cool-ratio-function f with the base XY . Then $f(M)f(N) = f(A)f(D)f(B)f(C) = f^2(P) = (-1)^2 = 1$, which is equivalent to the symmetry of M and N about the midpoint of XY . $\square$

Example (Partial case of the Pascal theorem). Let points A, B, C, D, E, F lie on a circle ω , and XY be a chord of ω , such that AD and BC , CF and DE meet on XY . Prove that AF and BE meet on XY .

Proof. Define the cool-ratio-function f with the base XY . Let AD and BC meet at M , and CF and DE meet at N . Then $f(A) \cdot f(F) = \frac{f(A) \cdot f(F) \cdot f(C) \cdot f(D)}{f(C) \cdot f(D)} = \frac{f(M) \cdot f(N)}{f(C) \cdot f(D)} = \frac{f(B) \cdot f(E) \cdot f(C) \cdot f(D)}{f(C) \cdot f(D)} = f(B) \cdot f(E)$. Therefore AF and BE also meet on XY . $\square$

What is CRL? Many people do not accept CRL, because it uses some relations between segments, which are not clear intuitively, and seems to be a pure calculative method. But we can remember, that the cross-ratio (from euclidean point of view) also uses the segments and the sines. Although this the cross-ratio is not considered as a non-synthetic geometry and is actively used in the problems. Why? The reason is that the cross-ratio can be generalized for projective planes; but the main — the cross-ratio is considered as an invariant with respect to the projections, the symmetries, and the projective maps. In the other words, we upset the cross-ratios and solve the problems using their properties, but not thinking that the segments and the sines are used in their definition. And we can understand CRL similarly!

Problem 1. Let $ABCD$ be a trapezoid with bases BC and AD inscribed into a circle ω . Let S be the midpoint of the arc AD , not containing B . A point E lie on the extension of AD beyond A . The segments CE and SE meet for the second time ω at points X and Y respectively. Prove that AD , SX , and BY concur.

Problem 2. A triangle ABC and points E, F on the sides AB and AC respectively are given. The point M is the projection of the circumcenter to EF , AM meets the circumcircle of ABC at point D . Let EF , BC , to the circumcircle at D concur. Prove that M is the midpoint of EF .

Problem 3. A triangle ABC and its bisector AD are given. Let P and Q lie on the circumcircle (ABC) in such a way that $\angle BPD = \angle DQC = 90^\circ$. The lines PQ and BC meet at point T . Prove

that the line AT touches the circle (ABC) .

Problem 4.

- (a) (The circumcevian ping-pong theorem) A chord XY of a circle ω is drawn. Points P , Q , and R are fixed on the line XY , and A is an arbitrary point of ω . Let AP meet ω for the second time at A_1 , A_1Q meet ω at A_2 , A_2R meet ω at A_3 . Prove that the common point of AA_3 and XY do not depend on A .
- (b) Let a circle and $2n$ lines passing through a point P , lying outside the circle be given. Choose an arbitrary point A_1 on the first line. Let the tangent from this point to the circle meet the second line at A_2 , the tangent from A_2 meet the third line at A_3 , etc (cyclically). Prove that if this process will close after k circulations (i.e. $A_{2nk+1} = A_1$), then this will be true for any other point A_1 .

2.2 CRL helps to prove the concyclicity

Consider now another subject.

Theorem 2.4 (Second cool ratio lemma). Let a circle ω and its chord XY be given. Points A and B lie on ω , and points C and D lie on the line XY . Then A, B, C, D are concyclic if and only if $f_{XY}(A) \cdot f_{XY}(B) = f_{XY}(C) \cdot f_{XY}(D)$.

Example. Let M, Q lie on the sideline BC , and P, N on the circumcircle of triangle ABC . It is known that AP and AQ , AM and AN are symmetric about the bisector of angle BAC . Prove that the lines PM and QN concur at the circumcircle, and their common point lies on the circumcircle of triangle AQM .

Proof. Define the cool-ratio-function f with the base BC . Let AM , AQ , and PM meet the circumcircle at points K , L , and X respectively. Then $f(X) = \frac{f(M)}{f(P)} = \frac{\frac{f(A)}{f(N)}}{\frac{f(A)}{f(Q)}} = \frac{f(Q)}{f(N)}$, from this we obtain that X lies on the line QN . But $f(X) = \frac{f(M)}{f(P)} = f(M) \cdot f(L) = \frac{f(M) \cdot f(Q)}{f(A)}$. Therefore $f(A) \cdot f(X) = f(M) \cdot f(Q)$, i.e. A, X, M, Q are concyclic. $\square$

And can we replace a circle+line into two circles? The following assertion gives a positive answer to this question.

Theorem 2.5 (Third cool ratio lemma). Let circles ω_1 and ω_2 meeting at points X and Y be given. Points A and B lie on ω_1 , and points C and D lie on ω_2 . Then A, B, C, D are concyclic if and only if $f_{XY}(A) \cdot f_{XY}(B) = f_{XY}(C) \cdot f_{XY}(D)$.

Now note the following interesting idea.

Statement 2.3. An inversion about a circle passing through X and Y conserves $|f_{XY}|$.

If the center of the inversion circle lies on XY , it coincides with the midpoint of this segment, and the inversion swaps points lying inside and outside the segment. This yields the following assertion:

Corollary 2.4. Points A and B are marked on the line XY . Let M be the midpoint of XY . Then the following assertions are equivalent:

- $f_{XY}(A) = -f_{XY}(B)$ (or $[X, Y; A, B] = -1$);
- $\overline{MA} \cdot \overline{MB} = MX^2$ (i.e. A and B are inverse about the circle with diameter XY).

Problem 5. Let Ω be the circumcircle of triangle ABC . An arbitrary line ℓ meets this circumcircle at different points X and Y . Let M be the Miquel point of lines AB , BC , AC , and ℓ . Denote $f = f_{XY}$. Prove that $f(M) = f(A) \cdot f(B) \cdot f(C)$.

Problem 6. Let $ABCD$ be an isosceles trapezoid. Some circle passing through A touches the base BC at point E . Let X be the second common point of this circle with the circumcircle of the trapezoid. The line XE meets the circumcircle at point Y , and the line DY meets BC at point F . Prove that $CF = BE$.

Problem 7. Two circles ω_1 and ω_2 meeting at points P and Q are given. Points A and B lie on PQ , points K, L lie on ω_1 , and points M, N lie on ω_2 . It is known that the lines AK, BL touch ω_1 , and the lines AM, BN touch ω_2 in such a way that exactly two of points K, L, M, N lie on each side from SY . Prove that K, L, M , and N are concyclic.

Problem 8. Let B, C, D, E, F be concyclic points. The lines DE and DF meet BC at points X and Y respectively. The circumcircles of triangles XYD and BCD meet for the second time at point Z . Prove that E, F , the reflection of D about the midpoint of BC , and the reflection of Z about BC are concyclic.

Problem 9. Let AA_1, BB_1 , and CC_1 be the altitudes of triangle ABC . The lines AC and A_1C_1 meet at point B_2 . The line passing through B_1 and parallel to A_1C_1 meets AB and BC at points X and Y respectively. Prove that X, Y, B_2 , and the midpoint of AC are concyclic.

Problem 10* (Oral Moscow geometric olympiad 2025 10.5). Let ABC be an acute-angled scalene triangle. The perpendicular bisector to BC and the sidelines AB, AC form a triangle Δ . Let X be the common point of the tangents at B and C to the circumcircle of ABC . The line AX meets this circle for the second time at point G , and M is the midpoint of BC . Prove that the circumcircles of the triangles XGM and Δ are tangent.

Problem 11*. The median AM of triangle ABC meets its circumcircle at point F . The inscribed parabola with focus F touches BC , AC , and AB at points P , Q , and R respectively. Prove that P , Q , R , M are concyclic.

2.3 Generalized Cool Ratio Lemma

Theorem 2.6 (Generalized Cool Ratio lemma). Let a circle ω and its chord XY be given. Points A , B , C , and D lie on ω , and points E , F lie on the line XY . Then A , B , C , D , E , F lie on a conic if and only if

$$f_{XY}(A) \cdot f_{XY}(B) \cdot f_{XY}(C) \cdot f_{XY}(D) = f_{XY}(E) \cdot f_{XY}(F).$$

Problem 12. Let a circle and a line meeting at two points be given. Some conic meets the circle at points X , Y , Z , and T , and meets the line at points A and B . The lines AX , AY , BZ , and BT meet the circle for the second time at points X' , Y' , Z' , and T' respectively. Prove that X' , Y' , Z' , T' , A , and B lie on a conic.

2.4 The liaison of CRL with classical metric theorems

Statement 2.5 (Liaison with spiral similarity). A triangle ABC and points E , F lying on its sidelines AB and AC respectively are given. Let M be the common point of the circumcircles of triangles ABC and AEF . Then $|f_{AB}(M)| = |f_{EF}(M)| = \frac{AE}{BF}$.

In the previous problems we fixed one chord and use it as the base for CRL. But sometimes it is useful to introduce several cool-ratio-functions and use them to reformulate the required assertion! We can take as example the Thales, Ceva, and Menelaos theorems. Later, if a triangle ABC is given, we recommend to use the designations: $p(T) = f_{BC}(T)$, $q(T) = f_{CA}(T)$, $r(T) = f_{AB}(T)$, in such a way that $p(A), q(B), r(C) > 0$ (i.e, all three functions are positive inside the triangle ABC .)

Statement 2.6 (The Thales theorem). Let a triangle ABC and points E , F lying on its sidelines AB and AC respectively be given. Then the lines EF and BC are parallel if and only if $r(E) \cdot q(F) = 1$.

Statement 2.7 (The Ceva theorem). Let D , E , and F be points lying on the sidelines BC , AB , and AC respectively of a triangle ABC . Then the line AD , BE , CF concur or are parallel if and only if $p(D) \cdot q(E) \cdot r(F) = -1$.

Statement 2.8 (The Menelaos theorem). Let points D , E , and F lie on the sidelines BC , AB , and AC respectively of a triangle ABC . Then D , E , F are collinear if and only if $p(D) \cdot q(E) \cdot r(F) = 1$.

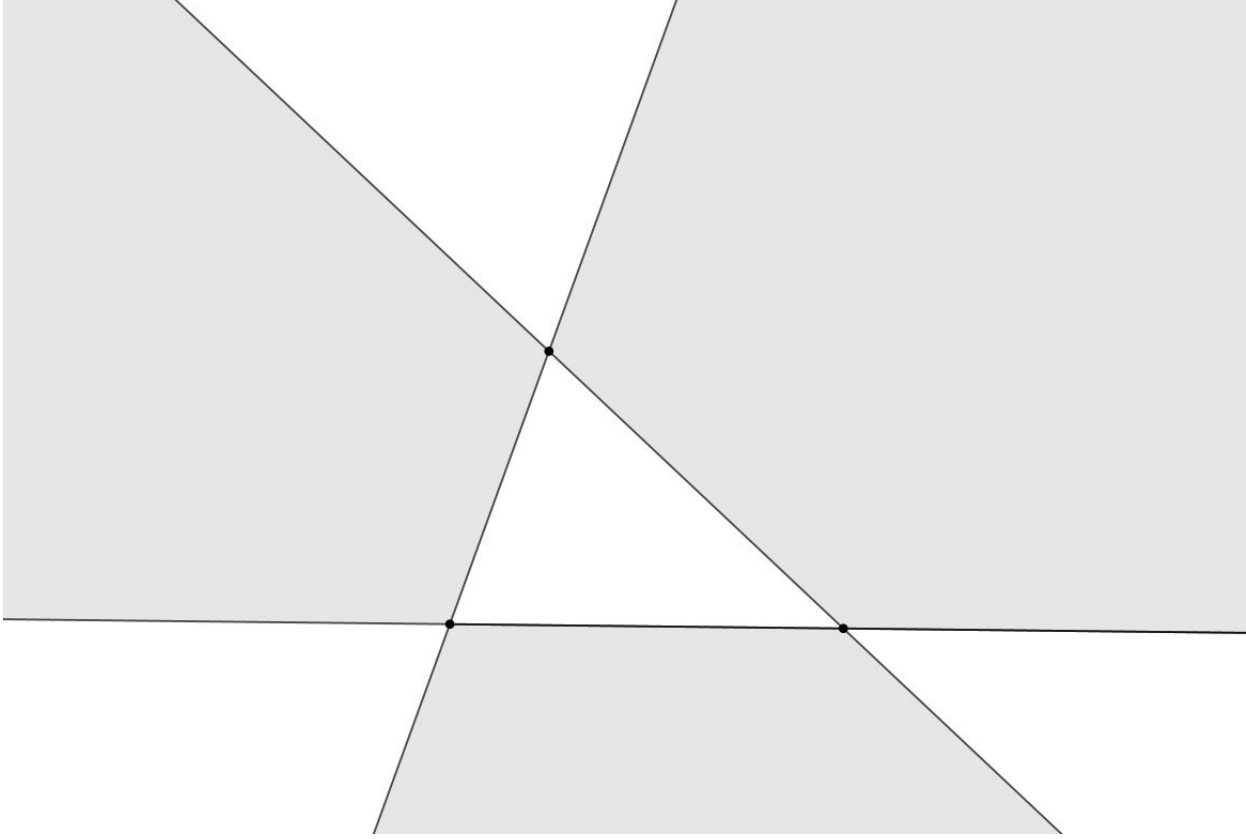
Statement 2.9 (The trigonometric Ceva theorem). Let a triangle ABC and points D , E , and F lying on (ABC) be given. Then the lines AD , BE , CF concur if and only if $p(D) \cdot q(E) \cdot r(F) = -1$.

Note also several simple but important properties:

Statement 2.10.

(a) $p(T) \cdot q(T) \cdot r(T) = \pm 1$, with a sign $+$ in gray sectors (see. the figure).

(b) $p(A) \cdot q(B) \cdot r(C) = 1$



Problem 13. A triangle ABC is given. The altitudes AA_1 , BB_1 , CC_1 meet the circumcircle for the second time at points A_2 , B_2 , C_2 respectively. The line B_2A_1 meets the circumcircle at X_1 , and B_2C_1 meets it at point Y_1 . The circumcircle of the triangle $A_1A_2X_1$ meets for the second time BC at point X_2 , and the circumcircle of the triangle $C_1C_2Y_1$ meets AB at Y_2 . Prove that $X_2Y_2 \parallel AC$.

Problem 14. Let D be an arbitrary point on the circumcircle of a triangle ABC , and M be the midpoint of BC . The circle AMD meets for the second time the line BC at point X . The line BD meets AC at point Y . The tangents to the circumcircle at B and C meet at point T . Prove that X , Y , and T are collinear.

Problem 15.

(a) Triangles ABC and DEF are inscribed into a circle Ω . The line BC meets DE and DF at points X and Y respectively. The circle (DXY) meets Ω for the second time at point D_1 . Points E_1 and F_1 are defined similarly. Prove that the lines AD_1 , BE_1 , and CF_1 concur.

(b*) (Difficult problem from JCG). The triangles ABC and $A'B'C'$ are inscribed into the same circumcircle, and the lines AA' , BB' , and CC' concur. Let A_1 be a common point of

(ABC) and the circle passing through A and the meeting points of $B'C'$ with AB, AC. Points $B_1, C_1, A'_1, B'_1, C'_1$ are defined similarly. Prove that the lines $A_1A'_1$, $B_1B'_1$, and $C_1C'_1$ concur.

Note another theorem which can be considered as analog of the theorem (2.6) for the case, when a circle+a line is replaced into three lines in general position.

Theorem 2.7 (The Carno theorem). Let a triangle ABC be given, points A_1, A_2 lie on BC, points B_1, B_2 lie on AC, points C_1, C_2 lie on AB. Then $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic if and only if

$$p(A_1) \cdot p(A_2) \cdot q(B_1) \cdot q(B_2) \cdot r(C_1) \cdot r(C_2) = 1.$$

Problem 16* (Moscow mathematical olympiad 2006, 10.6). Points P and Q lie on the circumcircle of a triangle ABC. Let A_1 be a point on the line BC such that the lines PA_1 and QA_1 are symmetric about BC. Points B_1 and C_1 are defined similarly. Prove that A_1, B_1 , and C_1 are collinear.

Problem 17. Let a triangle ABC be given, points A_1, A_2 lie on BC, points B_1, B_2 lie on AC, points C_1, C_2 lie on AB in such a way that $A_1A_2B_1B_2C_1C_2$ lie on a conic. Let $A_3, A_4, B_3, B_4, C_3, C_4$ lie on the circumcircle of ABC in such a way that $A_1A_2A_3A_4, B_1B_2B_3B_4, C_1C_2C_3C_4$ are concyclic quadruples. Prove that:

- (a) the triangle formed by the lines A_3A_4, B_3B_4 , and C_3C_4 , is perspective to ABC.
- (b) If AA_3, BB_3 , and CC_3 concur, then AA_4, BB_4 , and CC_4 also concur.

Compact record of ratios of segments is one of advantages of CRL, Hence we usually tend to translate the calculations into the cool-ratio-functions language. But sometimes to verify the final equality it is useful to return to the standard record of segments. In this way it is important not forget to verify the coincidence of the signs. The following problems are devoted to this subject.

Problem 18 (SMT¹-2019). Let AD, BE, and CF be concurrent cevians in a triangle ABC with circumcircle ω , and M be the midpoint of BC. The circles ω and (AEF) meet for the second time at point X. The line XD meets ω at point Y. The line YM meets ω at point Z. Prove that the line AZ touches (AEF).

Problem 19. A triangle ABC is given. Let D and E lie on the sidelines AB and BC respectively, in such a way that ADEC is cyclic. Let there exist a point T on the sideline AC, such that $BT = BD, CT = CE$. Prove that the circumcircle of the triangle DET passes through the foot of the internal bisector of angle B.

¹Southern Mathematical Tournament is the Russian team tournament of mathematical battles.

3 Euclidean invariants of the Poncelet theorem

This section is devoted to different invariants of the Poncelet theorem and the using of this theorem in the olympiad problems. the subsections are independent, thus it is not necessary to solve the problems in the proposed order. But it can be useful solving several problems to return back and scoop some useful idea.

Of course many problems are not original: they are taken from the papers and the olympiads including the recent ones. Hence you can see some problems or even know their solutions. This is not dreadful: there are many problems in this sections (may be too many), Thus you in all cases will not be bored.

The rules of the game:

- In all problems «the triangle rotates between two circles / conics» denotes the rotation in sense of the Poncelet theorem.
- «Inscribed conic» denotes that the conic touches the sidelines of the polygon — partially it can lie outside the triangle. But we differ the incircle and the excircle.
- If the existence of the circle having some properties has to be proved (a point moves around it, a variable circle touches it, etc.), the required circle can degenerate to a point or a line. Similarly, the assertion «the lines pass through a fixed point» allows that this point is infinite. And the term «inversion» may denote not only a usual inversion, but the composition of an inversion and the reflection about its center (so called «antiinversion»). This list can be prolonged. The general rule is: if it seems to you, that the formulating of some problem is not correct, turn to the authors.
- Some problems contain several points. The reasons to group several assertions to one problem can be different. Sometimes the previous points help to prove the following ones — see for example the proof of the Laguerre theorem. Or we join an assertion and its generalization (for example in problem (53)), different properties of a common construction, or similar assertions, for example in problem (45). As an exception the principle of joining can be not evident and will be clear later.

3.1 The Clawson conjugation and the Skutin lemma

Problem 20. Let a quadrilateral $ABCD$ rotate between a circumcircle Ω and an inconic γ with foci P and Q .

- Prove that the Clawson conjugation related with the quadrilateral is constant.
- Let the inconic be a circle, and AB and CD meet at point E . Prove that all circles (BCE) touche some fixed circle.
- Return to the case of an inellipse. Prove that the sum $\angle ABP + \angle CDP$ is constant.

The following interesting theorem can be proved using the Clawson conjugation, but it is not immediately coherent with the Poncelet theorem, hence we do not formulate it as a problem.

Theorem 3.1 (The Miquel theorem about isogonal conjugation). Let a triangle ABC and two pairs (P, P') , (Q, Q') of isogonally conjugated points be given. Then the Miquel point of the quadrilateral $PQP'Q'$ lies on the circumcircle of ABC .

This theorem is useful for the proof of the Skutin lemma — one of the most important results of this section.

Theorem 3.2 (The Skutin lemma). Let a triangle ABC rotate between a circumcircle Ω and an inellipse γ . A fixed point P does not lie on Ω . Then the point Q isogonally conjugated to P with respect to ABC moves around some circle.

Problem 21. Let F_1 and F_2 be the foci of the ellipse, and M be the Miquel point of F_1PF_2Q .

- (a) Record the map $M \mapsto Q$ in complex numbers and prove the Skutin lemma.
- (b) Consider a composition of the inversion with center P and radius $\sqrt{PF_1 \cdot PF_2}$ and the reflection about the bisector of angle F_1PF_2 and prove the Skutin lemma.

Considering with attention any of two proofs of the Skutin lemma we can describe the trajectory of the orthocenter:

Corollary 3.1. The orthocenter of the Poncelet triangle moves around the circle centered at the reflection of O about the center of inellipse with radius $\frac{OF_1 \cdot OF_2}{R_{(ABC)}}$.

Problem 22. Let the inconic be a circle with radius r . Prove that the Nagel point of ABC moves around the circle centered at O with radius $R - 2r$.

Problem 23. Let a triangle ABC rotate between two circles, and ℓ be a fixed tangent to the incircle.

- (a) Prove that the Miquel point of lines AB , BC , AC , and ℓ is constant.
- (b) Remember that the orthocenters of four triangles formed by four given lines are collinear on the *Aubert line*. Prove that the Aubert line of four lines from p.(a) passes through a fixed point.
- (c*) Let ω_H be the trajectory of the orthocenter of ABC . Prove that the fixed point obtained in the previous point lies on Ω_H .

Remark. Compare p.(b) with the problem 6 from the project «On the Poncelet theorem» from Summer Conference-2014:

Statement 3.2. Let ABC rotate between two circles, and P be a fixed point on the circumcircle. Prove that all Simson lines of P with respect to ABC pass through a fixed point.

Problem 24.

- (a) (Telv Cohl², 2014) A triangle ABC is given. Let ℓ be a variable tangent to its incircle. The line ℓ and their reflections about the sidelines of ABC form a complete quadrilateral. Prove that its Miquel point moves around a circle or a line.
- (b) Prove the same assertion when the line ℓ is fixed, and the triangle ABC rotates between two circles.

Problem 25. Let ABC rotate between two circles. A fixed line ℓ touching the incircle meets BC and AC at points D and E respectively. Let ℓ' be a line passing through the reflections of D and E about the midpoints of BC and AC respectively.

- (a*) (SAGF³ 2024, P3) Prove that ℓ' passes through a fixed point.
- (b*) Let ω_N be the trajectory of the Nagel point of ABC . Prove that the fixed point from the previous assertion lies on ω_N .

Problem 26.

- (a) (The Morley theorem) Let points P and Q be isogonally conjugated with respect to a triangle ABC . Choose a complex coordinate system such that (ABC) is a unit circle and denote the complex coordinates of all points by the corresponding small letters. Prove that

$$p + q + abc\overline{pq} = a + b + c.$$

- (b) Let a triangle Δ rotate between a circumcircle and an inconic, and P, Q be fixed points. By the Skutin lemma the points P', Q' isogonally conjugated to P and Q with respect to Δ move around two circles. Prove that their angle velocities are equal.

3.2 Liaison with polar correspondence

The following fact was proposed by E.Diomidov and V.Kalashnikov in 2014 to the readers of «Journal of Classical Geometry» as an open problem. As is known to the authors, the problem was not solved during a long time.

Statement 3.3. Let a triangle Δ rotate between its circumcircle Ω and (ex)incircle γ . Then the trilinear polar of a fixed point $P \in \Omega$ with respect to Δ passes through a fixed point.

²Telv Cohl is a known Taiwan geometer and problems compositor, author of blog [12].

³SAGF — geometric online-olympiad organized by K.Belsky. Four problems in each of two days, the complexity corresponds to the problems G3-G8 of IMO shortlist.

The short and nice proof was found by the authors less than one year ago. The following problem is the key one:

Problem 27. Let a triangle Δ rotate between a circumconic and an inparabola. Prove that the centroid of Δ moves along a line, if the circumconic is:

- (a) an ellipse;
- (b) a hyperbola;
- (c*) a parabola.

It is not difficult to prove that the problem of Diomidov and Calashnikov follows from the previous one (but you can solve it independently).

Problem 28. Prove the assertion (3.3) if P lies:

- (a) inside γ .
- (b) outside γ .
- (c*) on γ .
- (d**) It may be proved (see the calculation in barycentric coordinates in [5]), that the Lemoine point of ABC moves around an ellipse. Prove that the fixed point in the last case lies on this ellipse.

The following problem is the unique one in this section considering the complex projective plane. Namely, we will work with the limit points of a pencil of conics (the pentagon and the conics may be considered as real).

Problem 29* (F.Nilov). Let a pentagon $ABCDE$ be inscribed into a conic Γ and circumscribed around a conic γ . Its diagonals form a pentagon $A_1B_1C_1D_1E_1$ ($A_1 = BD \cap CE$ etc.), inscribed into a conic Γ_1 and circumscribed around a conic γ_1 . Prove that the pencils generated by pairs (Γ, γ) and (Γ_1, γ_1) have the same limit points.

3.3 The Laguerre theorem

Theorem 3.3 (The Euler formula). Let a triangle Δ be inscribed into a circle with center O and radius R and circumscribed around a circle with center I and radius r . Then

$$R^2 - OI^2 = 2Rr. \quad (3.1)$$

There is a natural question about the generalization of the Euler formula for an arbitrary inconic. The Laguerre theorem give the answer:

Theorem 3.4 (The Laguerre theorem, 1879). Let a triangle ABC be inscribed into a circle centered at O with radius R and circumscribed around an ellipse with foci F_1, F_2 and minor axis

l. Then

$$(R^2 - OF_1^2)(R^2 - OF_2^2) = R^2 l^2. \quad (3.2)$$

Remark. The theorem is also correct, if we replace an ellipse to a hyperbola, and consider l^2 as the square of the imaginary axis (equal to $l^2 = a^2 - d^2$, where d is the distance between the foci of the hyperbola, and a is the distance between the common points of the hyperbola with the line passing through its foci).

Remark. The Laguerre theorem yields a converse assertion: if the equality (3.2) holds, then there exists a triangle inscribed into the circle and circumscribed around the conic.

Prove now the Laguerre theorem.

Problem 30.

- (a) Let m be an arbitrary tangent to the ellipse with foci F_1, F_2 and minor axis $2b$. Then $\text{dist}(F_1, m) \cdot \text{dist}(F_2, m) = b^2$.
- (b) **General trident lemma.** Let L_1 and L_2 be the second points of the circumcircle (ABC) with the rays AF_1 and AF_2 . Prove that $F_1 L_1 \cdot F_2 L_2 = BL_1 \cdot BL_2$.
- (c) Obtain the Laguerre theorem from the pp. (a), (b).

Theorem 3.5 (3-Poncelet pairs theorem). Let spheres Ω_1 and Ω_2 touche a plane α on the different hands. A sphere Ω touches these two sphere by the same manner and meets α by a circle ω . The common external tangents to Ω_1 and Ω_2 meet α by an ellipse γ . Then there exists a triangle with circumcircle ω and inellipse γ .

Theorem 3.6 (The Grace theorem). Let ω and ω_A be the insphere and the A -exsphere of a tetrahedron $ABCD$. Then there exists a sphere passing through B, C, D and touching ω, ω_A .

Problem 31.

- (a*) Prove the 3-Poncelet pairs theorem.
- (b) Obtain the Grace theorem from the 3-Poncelet pairs theorem.

3.4 Latent Poncelet. Part 1

The using of the Poncelet theorem in olympiad problems do not necessary mean the process of rotation. Sometimes the configuration is static, and the Poncelet theorem appears in face of the following simple corollaries:

Corollary 3.4. Let ABC be a triangle with circumcircle Ω and incircle ω . Take an arbitrary point D on Ω . The tangents from D to ω (if they exist) meet Ω for the second time at E and F . Then EF touches ω .

Corollary 3.5. Let ABC be a triangle with circumcircle Ω and incircle ω . An arbitrary chord EF of Ω touches ω . Then the second tangents from E and F to ω meet on Ω .

Corollary 3.6. Let the triangles ABC and DEF are inscribed into the same conic. Then they are circumscribed around a common conic; in partial the main diagonals of the hexagon formed in the intersection of the triangles concur. Of course the converse assertion is also correct.

Problem 32. Denote by ω_a the excircle of triangle ABC touching the side BC . The circles (ABC) and ω_a meet at points X and Y . The tangents at X and Y to ω_a meet for the second time the circumcircle (ABC) at points P and Q . Prove that the tangents to (ABC) at points P and Q are also the tangents to ω_a .

Problem 33. Let I be the incenter of a triangle ABC , and X, Y be the points on its circumcircle such that $AI = AX = AY$. Prove that XY touches the incircle of ABC .

Problem 34 (Preparations, May 2022, P2/3). A cyclic quadrilateral $ABCD$ is given. Let ω_A and ω_C be the incircles of triangles DAB and BCD centered at I_A and I_C respectively. The common external tangent to ω_A and ω_C meets the circumcircle $(ABCD)$ at points X and Y . Prove that I_A , I_C , X , and Y are concyclic.

Problem 35 (IGO-2019, Advanced, P5). A parabola with focus F is given. Consider all triangles inscribed into the parabola having the orthocenter at F . Prove that all such triangles have a common circumcircle.

Problem 36. Let O and H be the circumcenter and the orthocenter of a triangle ABC . The line AH meets for the second time the circumcircle (ABC) at point A_1 , and the perpendicular bisector to AH meets (ABC) at points A_2 and A_3 . Points $B_1, B_2, B_3, C_1, C_2, C_3$ are defined similarly. Prove that the Lemoine points of the triangles $ABC, A_1A_2A_3, B_1B_2B_3$, and $C_1C_2C_3$ lie on a circle with the center on the line OH .

3.5 The latent Poncelet. Liaison with isogonal conjugation

For any pair of isogonally conjugated points there exists an inconic with foci at these points. Hence it may be useful to rotate the triangle between the circumcircle and this conic. The Skutin lemma and the Laguerre theorem also can be useful, especially if some other remarkable points are mentioned in the problem.

Let an ellipse with foci P and Q and the major axis $2a$ be given. Let a line ℓ touche it at point X , and P' be the reflection of P about ℓ . The optical property yields that P', X , are collinear Q , thus $P'Q = PX + QX = 2a$. From this we obtain a simple but very useful result:

Statement 3.7. Let an ellipse with foci P , Q and major axis $2a$ is inscribed into a triangle Δ . Then the radius of the pedal circle of P and Q with respect to Δ equals a .

Problem 37 (The third Fontené theorem). Let P and Q be isogonally conjugated with respect to a triangle ABC .

- (a) Prove that if PQ passes through the circumcenter of ABC , then the pedal circle of P and Q the nine-points circle of ABC .
- (b) Prove the converse assertion.

Remark. The locus of points P from the previous problem is called *the Mac-Key cubic*.

Problem 38 (SAGF-2024, P7, reformulated). Let a triangle ABC be inscribed into a circle centered at O with radius R and circumscribed around an ellipse centered at X with minor axis l . A point N is the center of the nine-point circle of ABC .

- (a) Prove that if the circumcenter O lies on the minor axis, then $2NX = R - l$.
- (b) Prove that in general case $2NX \leq R - l$.

3.5.1 The orthocenter power lemma

Problem 39 (The orthocenter power lemma). Let O and H be the circumcenter and the orthocenter of a triangle ABC . Points P and Q are isogonally conjugated with respect to ABC , and ρ is the radius of the pedal circle of P and Q . Then

$$4\rho^2 - HP^2 - HQ^2 = -\text{pow}(H, (ABC)).$$

In other words if P_a, P_b, P_c are the reflections of P about the sidelines of ABC , then

$$\text{pow}(H, (P_a P_b P_c)) + HP^2 = \text{pow}(H, (ABC)).$$

Corollary 3.8. Let P and Q be isogonally conjugated with respect to a triangle ABC . The points P_a, P_b, P_c are the reflections of P about BC, AC, AB . Define similarly Q_a, Q_b, Q_c . The orthocenter H of ABC lies on the radical axis of circles $(P_a P_b P_c)$ and $(Q_a Q_b Q_c)$ if and only if $HP = HQ$.

Problem 40.

- (a) Let triangles ABC and DEF have a common circumcenter and be circumscribed around the same parabola. Prove that their vertices lie on an equilateral hyperbola.
- (b) (Telv Cohl, 2019) Let A, B, C, D, E , and F lie on a conic which is **not** an equilateral hyperbola. It is known that the triangles ABC and DEF have a common orthocenter H . Prove that H lies on the radical axis of circles (ABC) and (DEF) .

3.5.2 The Aiyer theorem

Problem 41. Let a triangle ABC rotate between a circumcircle Ω and an inconic with foci P and Q . Let M_a be the Miquel point of $BPCQ$. It is easy to see that $M_a \in \Omega$. Prove that the map $A \mapsto M_a$ is a projective involution.

Corollary 3.9. All lines AM_a pass through a fixed point or are parallel.

Problem 42*. Let a triangle ABC rotate between a circumcircle and an inconic with foci P and Q . Prove that $\angle ABP + \angle BCP + \angle CAP = \text{const.}$

Problem 43 (The Aiyer theorem). A triangle ABC and a point P are given. Let $P_aP_bP_c$ be the pedal triangle of P , and Ω_9 be the nine-points circle of ABC . Prove that the directed angle between the tangents to the circles $(P_aP_bP_c)$ and Ω_9 at its common point equals $\pm(90^\circ - \angle ABP - \angle BCP - \angle CAP)$.

3.5.3 Problem 10.4 from the last Sharygin olympiad

Problem 44 (The Brocard porism). Let a triangle ABC rotate between a circumcircle and an inellipse with foci P and Q . Suppose that for one position P and Q are the Brocard points with angle φ . Prove that for all positions P and Q are also the Brocard points with angle φ .

Problem 45. Let M , H , O , and L be the centroid, the orthocenter, the circumcenter, and the Lemoine point of a triangle ABC .

(a*) (Sharygin olympiad 2025, final, 10.4) Let S be the L -Humpty point of the triangle LHM , and L^* be the inverse map of L about the circumcircle. Prove that $SL^* \parallel MH$.

(b*) Let T be the M -Humpty point of $\triangle LOM$, and M^* the inverse map of M about the circumcircle. Prove that $TM^* \parallel OL$.

Remark. The assertion (b) can be reformulated: the inverse map of M about the circle with diameter OL lies on the circumcircle of ABC .

3.6 Around classical proof of the Poncelet theorem

This subsection contains problems about the coaxial circles lemma and classical properties of the Poncelet polygons (see the details in subsection (1.7)).

Theorem 3.7. Let a natural number $2 \leq k \leq \lceil \frac{n}{2} \rceil$ be given. A polygon $A_1A_2 \dots A_n$ rotates between two circles. By the theorem (1.13) the line A_kA_{n+2-k} touches some fixed circle. Then the touching point projectively depends on A_1 .

Remark. Define in a natural way the cross-ratio of tangents to a conic. Then we can say that the line A_kA_{n+2-k} projectively depends on A_1 .

Corollary 3.10. Let a polygon $A_1A_2 \dots A_{2s+1}$ rotate between two circles. Then the touching point of the sideline $A_{s+1}A_{s+2}$ with the incircle projectively depends on A_1 .

Remark. The corollary (3.10) was proposed on the Summer conference in 2022 (project [2], problem 25). Any of participants could not solve it, and the authors of the project had then only a complicated algebraic proof. A simple synthetic proof which we propose to do was found later.

Problem 46.

- (a) (Generalized Poncelet homography). Let circles Ω and γ be given. The tangents from an arbitrary point $A \in \Omega$ (if they exist) to γ meet Ω for the second time at points B and C . By the theorem (1.13) all lines BC touche some fixed circle. Let K be the touching point. Prove that K projectively depend on A .
- (b) Prove the theorem (3.7).

Remark. The theorem (1.7) yields that the standard and the generalized Poncelet homographies hold also for conics having at most two common points. But we allow to use these assertions for an arbitrary pair of conics: you always may use the standard homography, and use the problem (46) — after solving it. If you want to know the general proof of these facts we will give it in the additional section.

Problem 47 (D. Brodskiy). Let a circle ω lie inside a circle Γ . Let $X \in \omega, Y \in \Gamma$ be two cyclists starting from two closest points of ω, Γ and moving along these circles in the same direction with equal angle velocities. Let P, Q be points on ω such that $PX = QX = XY$. Prove that PQ touches a fixed circle.

Problem 48. A circle ω lies inside a circle Ω . Consider a closed chain of circles $\gamma_1, \gamma_2, \dots, \gamma_{2n}$ such that γ_i centered at O_i touches externally ω , touches internally Ω , and touches $\gamma_{i-1}, \gamma_{i+1}$ (as usual, $\gamma_{i \pm 2n} = \gamma_i$). Prove that all lines O_iO_{i+n} concur, and their common point depend only on ω and Ω , but not on the chain.

Problem 49 (SMT 2024, generalization). A polygon $A_1A_2 \dots A_{2n}$ rotates between two circles. Prove that the value $\frac{\sum_{i=1}^n A_iA_{i+n}}{S_{A_1 \dots A_{2n}}}$ is constant.

Problem 50.

- (a*) A polygon $A_1A_2 \dots A_n$ rotates between two circles. The line A_iA_{i+1} touches the incircle at point X_i . Prove that the mass-center of $X_1, X_2, \dots, X_n$ is fixed.
- (b) A polygon $A_1A_2 \dots A_n$ rotates between two circles, and a point P is fixed. The line A_iA_{i+1} touches the incircle at point X_i . Prove that $\sum_{i=1}^n PX_i^2$ is constant.
- (c) Let now $n = 2m$. Prove that $\sum_{i=1}^{2m} \frac{A_iA_{i+1}}{A_{m+i}A_{m+i+1}}$ is constant.

The (im)possibility of construction of bicentral polygons using a compass and a ruler is very interesting. We practically do not touch it in the project and give only the following problem, which yields that it is sufficient to construct polygons with odd number of sides

Problem 51. A circle β lies inside a circle α . It is known that there exists a n -gon inscribed into α and circumscribed around β . Using a compass and a ruler construct a $2n$ -gon inscribed into α and circumscribed around some circle β' coaxial with α and β .

3.7 Different

We do not tend to the maximal generality of results in this section. In many problems a circle may be replaced to an arbitrary conic with two foci. If the difference seems essential, we formulate both version (see for example the problem (53)). But often the solution is practically the same (of course if your solution is appropriate). We propose to you to verify this.

Problem 52. Solve the problems (23), (24), (25), (28), (32) и (56a) if the (ex)incircle is replaced to a conic with two foci.

Problem 53. Let a triangle ABC rotate between a circumcircle and an inscribed

(a) (Sharygin olympiad, 2023, final, 10.4) circle

(b*) (Xiuyi Chen, aka LoloChen, 2024) conic

centered at X . Let A_1 be the reflection of A about X . Prove that the point isogonally conjugated to A_1 with respect to ABC moves along a line.

The following problem is in some sense the converse to the Skutin lemma.

Problem 54* (Xinchen Yu, 2024). A triangle ABC rotates between a circumcircle and an inconic centered at X . Let G be the centroid of ABC , and a point Y be such that $\overrightarrow{GY} = -2\overrightarrow{GX}$. Prove that Y is isogonally conjugated to a fixed point.

Remark. If the conic is the incircle, then Y is the Nagel point isogonally conjugated to the center of the positive homothety of the incircle and the circumcircle, which is clearly fixed.

Problem 55.** It is easy to see the similarity of the problems (23), (25), and (28). Formulate a general assertion including them.

Problem 56.

(a*) A hexagon $ABCDEF$ rotates between two circles. Let P be a fixed point, and points Q , R be isogonally conjugated to P with respect to the triangles ACE and BDF respectively. Prove that all lines QR are concurrent or parallel.

(b**) Let a hexagon $ABCDEF$ rotate between two arbitrary conics. The circumcircles of triangles ACE and BDF meet for the fourth time the circumconic at points X and Y . Prove

that all lines XY are concurrent or parallel.

We will call a family of circles a *copula*, if the centers of all these circles are collinear, or there exists a point having the same power with respect all circles.

Problem 57. Let the triangle Δ rotate between two arbitrary conics. Prove that its
 (a*) circumcircle
 (b**) nine-point circle
 lies on a fixed copula.

3.7.1 The Poncelet broken lines

The inclosed Poncelet broken lines were educated by the geometers. For example see the problems 7 and 10 from the previous project about Poncelet. In this subsection we educate them more detailly and prove several assertions generalizing the known facts concerning the Poncelet polygons.

Formulate firstly a simple assertion which is interesting itself and can be useful in several problems of this subsection.

Statement 3.11. Let ℓ be the radical axis of circles ω_1, ω_2 centered at O_1 and O_2 respectively. Prove that $|\text{pow}_{\omega_1}(P) - \text{pow}_{\omega_2}(P)| = 2O_1O_2 \cdot \text{dist}(P, \ell)$.

We do not give the proof, you can obtain it projecting the point to the centerline and counting the segments.

Return to the subject. Consider firstly the broken lines with one link

Problem 58. Let AB be a moving chord of a fixed circle ω centered at O , and P be a fixed point inside ω . Prove that the following assertion are equivalent:

- The line AB touches a fixed conic with focus P .
- The circumcenter O_1 of (PAB) moves along a circle or a line.
- The projection of P to AB moves along a circle or a line.

Problem 59. An inclosed three-links broken line rotate between a circumcircle and an inconic. Let ω be the circumcircle of the triangle formed by the links of the broken line.

- (a) Suppose that the inconic is a circle. prove that ω touches two fixed circles.
- (b*) Prove that in general case ω lies in a fixed copula.

Remark. This problem generalizes the points (a) and (b) of the problem (20). The authors do not known is it possible that the circles from p.(a) degenerate to lines.

Problem 60. A circle Ω and a conic with foci P and Q are given. The tangents from an arbitrary point $A \in \Omega$ to the conic (if they exist) meet for the second time Ω at points B and C .

- (a) Prove that the point isogonally conjugated to P with respect to ABC moves along a circle or a line.
- (b*) Prove that the pedal circle of P with respect to ABC lies on a fixed copula.
- (c*) Let M be the Miquel point of $BPCQ$. Prove that there exists a fixed inversion or symmetry mapping A to M .

Remark. The p.(c) yields that M projectively depends on A and moves along a circle or a line, and all lines AM concur. Thus this problem generalizes the problem (41).

Problem 61. An unclosed three-links broken line $ABCD$ rotates between two circles. Let ℓ be the second common external tangent to the incircles of triangles ABC and BCD .

- (a*) Prove that the common point of ℓ and BC moves along a line.
- (b*) (G.Zabaznov) Prove that ℓ touches a fixed circle or passes through a fixed point.

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