

Linear characteristics of graphs

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Part I

We have a graph (without loops and multiple edges, probably disconnected). The following three parameters (but not only them) we call linear characteristics of the graph.

- (a) the number of vertices — n ,
- (b) the number of edges — m ,
- (c) independence number — α (the maximal size of the independent set).

It is interesting what values these linear characteristics may attain? Is it possible by the triple (n, m, α) to understand does such graph exist or not? It is famous problem, but it has interesting variations including recent results and even open problems.

One of the directions is to put this question for some family of graph, for example, for planar graphs, or distance graphs, or graphs that do not contain certain subgraphs. These problems have several common features, for example, with problems about the chromatic number of the plane, Borsuk's problem, and even with the packing of spheres. But best of all, probably, you are familiar with placing non-attacking chess pieces on an rectangular board.

Let a rectangular board and some chess piece X be given. We treat the cells of the board as the vertices of the graph, and all the possible moves of chess piece X as the edges of the graph. The constructed graph we call *chess* graph. For example, for the board 2×2 and the king it will be the complete graph K_4 , and for the board 2×2 and the rook it will be the cycle on 4 vertices C_4 .

The conception of superfluous edge

Edge is called *superfluous*, if its removal does not change the independence number of the graph. Graph containing no superfluous edges, is called α -critical, and any α -critical graph obtained from a graph G with independence number α by consecutive discarding superfluous edges, is called a *nucleus* of graph G . Connected α -critical graph we call *Turán's* graph.

Problem 1 Find some nucleus of the chess graph for:

- 1) rectangular board $m \times n$ and the rook,
- 2) board 3×3 and the queen,
- 3) cylindrical board $m \times n$ (the first column is glued with the last) and the king.
- 4) The chess piece cheetah from arbitrary cell x attacks all the cells of square 19×19 with the central cell x excluding the cells that are situated in the same column or in the same row with x . What largest number of not attacking each other cheetahs may be placed on the board 1000×1000 ? Find some nucleus of chess board for the cheetah and this board. (Russian olympiad, Regional round, 2018, 10.8.)

- 5) Draw some board connected relatively to the king move on the grid, on which cannot be placed 4 non-attacking each others kings and which cannot be covered by three squares 2×2 . Find the nucleus of the corresponding chess graph.
- 6) The vertices of graph G are the points of the plane (x, y) , $0 < x, y < 100$, where one of the numbers x, y is integer, and the second is half-integer. (The vertex of the graph may be interpreted as the center of a domino.) An edge connects two vertices, if the distance between them is equal to 1 (and the half-integer coordinates must coincide and the integer coordinates must differ by 1), or if the distance between the vertices is equal to $\sqrt{2}/2$. Find the independence number and some nucleus of this graph.

The graphs from problem 1 show us that erasing of superfluous edges in natural constructions often leads to disconnected nucleus consisting of several cliques. Thus natural question arises: for which n and α does there exists at least one Turán's graph at all?

Problem 2

- 1) For any positive integer α construct Turán's graph with $n = 2\alpha + 1$.
- 2*) For Turán's graph prove the inequality $n \geq 2\alpha + 1$.

Instruction. Probably, you cannot solve item 2 at once. But you may return to it in the part II using more powerful technique.

It will be useful to add vertices to the graph for constructing another examples. *Cloning of vertex* x is the adding of new vertex connected with x and all its neighbours.

Problem 3

- 1) Prove that the cloning of the vertex does not change the independence number α .
- 2) Prove that the cloning of the vertex of α -critical graph preserves it α -critical.
- 3) Give an example of Turán's graph with given α and $n \geq 2\alpha + 1$.

Thus, we have a method of correct adding vertices to the graph looking after their degree.

Problem 4 What is the largest number of vertices of graph with independence number α , in which

- 1) degree of any vertex does not exceed k ?
- 2*) degree of any vertex does not exceed k , and which does not contain cliques on $k + 1$ vertices?

Problem 5 (Turán's theorem)

Let $k = \lfloor \frac{n}{\alpha} \rfloor$. Denote

$$T(n, \alpha) = k \cdot n - \frac{k(k+1)}{2} \alpha.$$

- 1) Prove that the smallest number of edges of graph on n vertices with independence number α is equal to $T(n, \alpha)$.
- 2*) Prove that the smallest number of edges of CONNECTED graph on n vertices with independence number α is equal to $T(n, \alpha) + \alpha - 1$

- 3*) Prove that the smallest number of edges of graph on n vertices with independence number α and p components of connectivity is equal to $T(n, \alpha) + \alpha - p$
- 4*) Prove that the smallest number of edges of TURÁN'S graph on n vertices with independence number α is equal to $T(n + \alpha - 1, \alpha) - \alpha + 1$.

Hints. Try in the item 1) to prove estimation by induction comparing the number of edges in two graphs and on each step discarding one vertex of the largest degree from each of them. To solve items 2) and 3), it is useful to discard superfluous edges if they exist. Problem 6 helps to seek for superfluous edges. Item 4) is hard enough, but the ideas, explored in the second and third parts of the project, leads to its solution.

It would seem what is there a difference between items 2) and 4)? The situation where the prohibition of superfluous edges strengthens any lower bound on the number of edges, seems paradoxical at first.

Problem 6 Prove that Turán's graph does not have leaves and cut vertices (the vertices, removal of which increases the number of connectivity components).

Problem 7 Prove some lower bound for the number of graph edges that is correct for problem 5.4 and incorrect for problem 5.2. You can prove only case $\alpha = 2$, $n = 7$.

New characteristics and correct graph modifications

We know that nucleus of graph G is not uniquely defined. The largest possible number of connectivity components of nucleus we call *nuclear number* of graph G and denote by $p(G)$ or simply p . Update the list of linear characteristics.

- (a) the number of vertices — n ,
- (b) the number of edges — m ,
- (c) independence number — α ,
- (d) nuclear number — p .

Surprisingly, but the addition of the fourth characteristics helps give non-trivial bounds on the first three!

To prove lower bounds on the number of edges in α -critical graph (first of all in problem 5.4) and construct examples of such graphs, we need different graph modification, i. e. adding/removing of the vertices preserving the required characteristics. To add vertices we have already been able. Now we try to learn how correctly to remove vertices from the graph. Vertex x of α -critical graph G is called *stable*, if nuclear number of graph $G \setminus x$ is equal to nuclear number of G . (By $G \setminus x$ we denote the graph obtained from graph G by removing vertex x and all the edges incident to it.)

Problem 8 Assume that in Turán's graph G with $\alpha > 1$ all the vertices are stable.

- 1) Let graph G be not regular. How to construct Turán's graph with the same parameters n , α and the number of edges that is smaller than the previous?
- 2) Let graph G be regular. How to construct irregular Turán's graph with the same n , α and the number of edges that is not larger than the previous?

Thus, the solution of problem 8 allows to modify graphs with stable vertices preserving n , α , p and does not increasing the number of edges m . But what to do, if we want to remove a vertex of the graph, but this vertex is unstable? Call modification of α -critical graph *correct*, if for some $t \in \mathbb{Z}$ the following four conditions hold:

- 1) Independence number is saved.
- 2) Graph remains α -critical.
- 3) The number of vertices n and the number of connectivity components p will be equal to $n + t$ and $p + t$ respectively.
- 4) The number of edges of modified graph is no more than $m + t$.

For example, the modifications from the previous problem are correct for $t = 0$.

Problem 9 Invent a correct modification for $t = -1$, which works for graph consisting of two connectivity components G_1 and G_2 , each of them contains at least three vertices, where

- 1) G_1 and G_2 are cliques (for example, construct two different Turán's graphs with $n = 8$, $\alpha = 2$, $m = 15$ from the clique on 4 vertices and the clique on 5 vertices);
- 2) G_1 is Turán's graph, G_2 is clique;
- 3) G_1 and G_2 are Turán's graphs.

Correct modification with $t > 0$ we call a *split*.

Theorem 1 (Theorem about split). *Let x be an unstable vertex of Turán's graph G . Then for each connectivity component of nucleus of graph $G \setminus x$ one vertex may be added in such way that the obtained result is the correct modification of graph G .*

Theorem 2 (Main theorem). *Let G be arbitrary α -critical graph. Then there exists a correct modification that converts G into disjoint union of cliques.*

Definition. Associate Turán's graph for α -critical graph G on n vertices with independence number α and p components of connectivity is disjoint union of α cliques that contain exactly $n + \alpha - p$ vertices in total and the difference between sizes of any two of them is no more than 1.

Note that main theorem allows us to reorganize any Turán's graph into its associate Turán's graph by correct modification. From the other hand, multiply applied modification from problem 9 is the correct modification that converts disjoint union of cliques, containing at least 3 vertices each, into Turán's graph. Turán's graph, that has been obtained in such a way, we call *trivial*. Therefore, the bound on the number of edges

$$m \geq T(n + \alpha - p, \alpha) - \alpha + p,$$

that follows from main theorem, is sharp. For example, it turns into equality on trivial graphs, obtained by series of modifications from problem 9 from disjoint union of cliques, the difference between sizes of each pair of which is no more than 1. α -critical graph with a minimal possible (for given n , p , α) number of edges $m = T(n + \alpha - p, \alpha) - \alpha + p$ is called a *zero cost* graph.

Problem 10

- 1) Draw 4 different trivial Turán's graphs with $n = 10$, $\alpha = 3$, $m = 16$.
- 2*) In what case the series of modifications from problem 9.3 may be done in arbitrary order?
- 3**) Invent convenient classification of trivial Turán's graphs with given n and α .

Problem 11 Obtain the main theorem from the theorem about split.

Theorem 3 (About zero cost graphs). *Let G be non trivial zero cost graph. Then its associate Turán's graph contains a connectivity component containing exactly 3 vertices.*

Problem 12 4α points are marked on the plane, after that we connect by a segment each pair of points at distance 1 cm. It turns out that among any $\alpha + 1$ points there necessary exist the two connected by segment.

- 1) Prove that at least 7α segments have been drawn in total. (A.M. Raigorodskii, Moscow Math Olympiad 2010, 10.6)
- 2) Improve the bound to 7.5α segments.
- 3) Let the number of points equals to n , and the number of the segments equals to m . Prove bound $m \geq 5n - 12\alpha$.

Instruction. You can use without proof main theorem and the theorem about split. If the bound from item 3 seems to you too simple, and you have desire to essentially improve it (for example, to 9α), then we can get supplement problems on this theme. The theorem about zero costs graphs helps in these bounds. That is the end of part I. Further you can proceed to part II or part III. Each part leads to the proof of the theorem about split, but contain also other results. Part III also leads to the proof of theorem about zero cost graphs.