

Diophantine equations

Project is prepared and presented by:

I. Klimanova, P. Kozlov, A. Miroshnikov, E. Veretennikov.

Abstract

In this project you can get acquainted with some remarkable equations and problems arising from the theory of Diophantine equations. This project consists of several sections and it is highly recommended to solve problems step by step.

In section 1 we consider the Diophantine equation $x^2 - Dy^2 = n$ and we suggest that you are already familiar with the structure of solutions of equation $x^2 - Dy^2 = 1$. The theory of continued fractions helps to solve the given equation with a small n . Also, we provide a relatively little-known method that reduces equation $x^2 - Dy^2 = n$ to another one $t^2 - Du^2 = m$ with a smaller m .

Section 2 is devoted to special cases of the equations $x^4 - Dy^2 = 1$ and $x^2 - Dy^4 = 1$. While solving problems in this section, you will learn the structure of solutions to some equations $x^2 - Dy^2 = n$ with a small n . Understanding this structure is also quite useful for many other problems.

In section 3 we show the connection between the old problem on existence of Diophantine quadruples and the number of classes of solutions to equation $x^2 - (k^2 + 1)y^2 = k^2$.

1 On the diophantine equation $x^2 - Dy^2 = n$

Let α be an irrational number. We can represent α as an infinite continued fraction $[a_0; a_1, a_2, \dots]$. A continued fraction formed from $[a_0; a_1, a_2, \dots]$ by neglecting all terms after a given term is called a *convergent* to α . We can represent k -th convergent $[a_0, \dots, a_k]$ as an irreducible fraction with a numerator p_k and a denominator q_k . Denote a continued fraction by $[a_k, a_{k+1}, \dots]$ by α_k .

Properties of convergents:

- a) $p_k = a_k p_{k-1} + p_{k-2}$, $q_k = a_k q_{k-1} + q_{k-2}$.
- b) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$, $p_k q_{k-2} - p_{k-2} q_k = (-1)^k a_k$.
- c) p_{2k}/q_{2k} is an increasing sequence, p_{2k-1}/q_{2k-1} is a decreasing sequence.
- d) $\alpha = \frac{\alpha_{k+1} p_k + p_{k-1}}{\alpha_{k+1} q_k + q_{k-1}}$.

Problem 1.1. a) Prove the equalities:

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}, \quad p_k q_{k-2} - p_{k-2} q_k = (-1)^k a_k.$$

b) Prove the inequalities:

$$\frac{1}{q_{k+1} + q_k} < |q_k \alpha - p_k| < \frac{1}{q_{k+1}}.$$

c) Prove that a sequence $\{|q_k \alpha - p_k|\}_{k=0}^{\infty}$ is strictly decreasing.

Problem 1.2. Prove that of any two consecutive convergents to α , at least one satisfies the inequality

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{2b^2}.$$

Problem 1.3. Prove that of any three consecutive convergents to α , at least one satisfies the inequality

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{\sqrt{5}b^2}.$$

Problem 1.4. Consider a positive integer k and integers p, q with $q > 0$ such that

$$|q\alpha - p| < |q_k\alpha - p_k|.$$

a) Prove that there are integers u, v such that

$$p_k u + p_{k+1} v = p, \quad q_k u + q_{k+1} v = q.$$

b) Prove that $q \geq q_{k+1}$.

Using the previous problem one can show that the convergents form a sequence of «best» approximations to α .

Problem 1.5. If a/b is a rational number such that

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{2b^2}$$

then a/b is one of the convergents to α .

Further, throughout the project, D is a squarefree positive integer number.

Problem 1.6. Let n be an integer that is less than $\sqrt{D}$ in absolute value. Assume that equation $x^2 - Dy^2 = n$ has a solution x_0, y_0 in positive integers. Prove that x_0/y_0 is one of convergents to $\sqrt{D}$ if:

- a) n is positive;
- b) n is negative.

Definition. A real irrational number which is a root of a quadratic equation with integral coefficients is called a *quadratic irrational*.

An infinite continued fraction is said to be *periodic* if it is of form $[a_0; a_1, \dots, a_l, \overline{b_1, \dots, b_m}]$. If it is of the form $[\overline{b_1, \dots, b_m}]$ it is said to be *purely periodic*. The smallest positive integer m satisfying the above relationship is called the *period* of the expansion.

Problem 1.7. a) Prove that if α is a quadratic irrational and a, b, c, d are rational with not both c and d zero then number

$$\beta = \frac{a + b\alpha}{c + d\alpha}$$

is either rational or a quadratic irrational.

b) Prove that each periodic continued fraction is a quadratic irrational.

The converse to previous problem is also true.

Problem 1.8. a) Suppose α is a real, irrational root of $ax^2 + bx + c = 0$, where a, b, c are integers and $a \neq 0$. Prove that α_{k+1} is a root of the polynomial

$$A_k x^2 + B_k x + C_k = 0,$$

where

$$A_k = ap_k^2 + bp_k q_k + cq_k^2,$$

$$B_k = 2ap_k p_{k-1} + b(p_k q_{k-1} + p_{k-1} q_k) + 2cq_k q_{k-1},$$

$$C_k = ap_{k-1}^2 + bp_{k-1} q_{k-1} + cq_{k-1}^2$$

and

$$B_k^2 - 4A_k C_k = b^2 - 4ac.$$

b) Prove that there is a number M such that $|A_k|, |B_k|, |C_k| < M$ for any positive integer k .

c) Prove that each quadratic irrational has a periodic expansion as a continued fraction.

Definition. Let $\alpha = a + b\sqrt{D}$, where a, b are rational and $\sqrt{D}$ is irrational. The *conjugate* of α , denoted by α' , is the number $a - b\sqrt{D}$.

Problem 1.9. Let α be a quadratic irrational.

a) Prove that if the continued fraction of α is purely periodic then $\alpha > 1$ and $-1 < \alpha' < 0$.

b) Prove that if $\alpha > 1$ and $-1 < \alpha' < 0$ then the continued fraction of α is purely periodic.

Problem 1.10. Write the continued fraction $[a_0; a_1, a_2, \dots, a_k, \alpha_{k+1}]$ of $\sqrt{D}$.

a) Prove that α_k is purely periodic for any positive integer k .

b) Prove that there exist positive integers b_k, c_k such that

$$\alpha_{k+1} = \frac{b_k + \sqrt{D}}{c_k}$$

and

$$p_k^2 - Dq_k^2 = (-1)^{k+1} c_k.$$

It follows from the previous problem that sequence $p_k^2 - Dq_k^2$ is periodic with a period equal to or twice as large as the period of continued fraction for $\sqrt{D}$. This fact together with result of Problem 1.6 describes all the solutions to equation $x^2 - Dy^2 = n$ with $|n| < \sqrt{D}$.

In the sequel, we consider only solutions to equation $x^2 - Dy^2 = n$ with $x + y\sqrt{D} > 0$, because $(-x, -y)$ is also a solution whenever (x, y) is a solution.

It is easy to see that if the equation $x^2 - Dy^2 = n$ has an integer solution (x_0, y_0) then it has an infinite number of such solutions (x_k, y_k) defined by

$$x_k + y_k\sqrt{D} = (x_0 + y_0\sqrt{D})(s + t\sqrt{D})^k,$$

where (s, t) is the minimal positive solution to the corresponding Pell's equation.

Definition. Two solutions (x_1, y_1) and (x_2, y_2) to equation $x^2 - Dy^2 = n$ are said to be *in the same class* if there is an integer n such that

$$x_2 + y_2\sqrt{D} = (x_1 + y_1\sqrt{D})(s + t\sqrt{D})^k,$$

where (s, t) is the minimal positive solution to the corresponding Pell's equation.

We say that (x, y) is the *minimal* solution in the class if $1 < x + y\sqrt{D} \leq s + t\sqrt{D}$.

Problem 1.11. How many classes of solutions does an equation

$$x^2 - (m^2 - 2)y^2 = 2$$

have for any integer $m > 1$?

Definition. A solution (x_0, y_0) of $x^2 - Dy^2 = n$ is said to be *primitive* if x_0 and y_0 are relatively prime.

It is obviously sufficient to find only primitive solutions, because if $\gcd(x_0, y_0) = d > 1$ then $(x_0/d, y_0/d)$ is a primitive solution to

$$x^2 - Dy^2 = \frac{n}{d^2}.$$

Problem 1.12. If (x_0, y_0) is a primitive solution to equation $x^2 - Dy^2 = n$ then there exists an integer k such that

$$x_0 \equiv ky_0 \pmod{n}$$

and

$$k^2 \equiv D \pmod{n}.$$

It follows from the previous problem that if D is a quadratic nonresidue modulo n then the Diophantine equation

$$x^2 - Dy^2 = n$$

has no primitive solutions.

Definition. We say that a primitive solution (x_0, y_0) to the equation $x^2 - Dy^2 = n$ *belongs to the integer k* if

$$x_0 \equiv ky_0 \pmod{n}$$

and

$$k^2 \equiv D \pmod{n}.$$

Problem 1.13. Prove that two primitive solutions to equation $x^2 - Dy^2 = n$ are in the same class if and only if they belong to the same integer.

It follows from the previous problem that there exists only a finite number of classes for any Diophantine equation $x^2 - Dy^2 = n$.

Let (x_0, y_0) be a solution to equation $x^2 - Dy^2 = n$ that belongs to some integer k . It follows from definition that there are integers u_0, m such that

$$x_0 = y_0 k - u_0 n, \quad D = k^2 - mn.$$

Problem 1.14. Prove that there is an integer t_0 such that (t_0, u_0) is a solution to equation $t^2 - Du^2 = m$ belonging to the integer $\pm k$. Express x_0, y_0 in terms of t_0, u_0 .

The number m from the previous problem depends on k . Notice that the choice of k is not fixed, it is fixed only modulo n .

Problem 1.15. If $|n| > \sqrt{D}$, the number m (from the previous problem) can be chosen to satisfy inequality $|m| \leq \max\{\sqrt{D}, |n|/4\}$.

By applying the method from the previous two problems repeatedly, we eventually obtain equations

$$t'^2 - Du'^2 = m',$$

where $|m'| < \sqrt{D}$, which have only those integer solutions that we obtain from the numerators and denominators of convergents to $\sqrt{D}$. A word of caution: in applying the method repeatedly we can pick different values of k at each step.

Problem 1.16. Determine all the classes of solutions to the following equations:

- a) $x^2 - 82y^2 = 23$,
- b) $x^2 - 23y^2 = 41$.

Problem 1.17. Prove that equation $x^2 - 229y^2 = 3$

- a) has no integer solutions,
- b) has a solution modulo every prime p .

2 On the equations $x^4 - Dy^2 = 1$ and $x^2 - Dy^4 = 1$

To solve the problems in this section, it is helpful to solve the problems from the first section, though not mandatory. To understand the formulation of some problems, it is necessary to get acquainted with the definition of «class of solutions» and «minimal solution» in the class of solutions to equation $x^2 - Dy^2 = n$ from the first section (before Problem 1.11).

Definition. Suppose equation $x^2 - Dy^2 = n$ has a single class of solutions. We say that a number $\alpha = u + v\sqrt{D} > 0$ generates the fundamental solution to $x^2 - Dy^2 = n$ if (u, v) is the minimal solution in this class.

Problem 2.1. Prove that if equation $x^2 - Dy^2 = -1$ has a solution then it has a single class of solutions. Prove that if $\alpha = u + v\sqrt{D}$ generates the fundamental solution to $x^2 - Dy^2 = -1$ then the fundamental solution to $x^2 - Dy^2 = 1$ is given by $\beta = \alpha^2$.

Problem 2.2. Let (u, v) be the fundamental solution to Pell's equation $x^2 - Dy^2 = 1$. Prove that equation $x^2 - Dy^2 = -1$ has a solution if and only if $u \equiv -1 \pmod{2D}$.

Problem 2.3. a) Consider coprime positive integers $m, n > 1$ such that mn is not a perfect square and equation $mx^2 - ny^2 = 1$ has a solution. Let (u, v) be the minimal positive solution to this equation. Prove that the fundamental solution to $x^2 - mny^2 = 1$ is given by $\gamma = \alpha^2$, where $\alpha = u\sqrt{m} + v\sqrt{n}$.

b) For any integer $a \geq 2$, describe all the integer solutions to $ax^2 - (a-1)y^2 = 1$.

Problem 2.4. Suppose that the fundamental solution to $x^2 - Dy^2 = -4$ is $2\alpha = a + b\sqrt{D}$, where a and b are odd integers. Find the fundamental solutions to equations

$$x^2 - Dy^2 = 4, \quad x^2 - Dy^2 = -1, \quad x^2 - Dy^2 = 1.$$

Problem 2.5. Prove that the following equations have no solutions in positive integers:

a) $x^4 - 8z^4 = 1$; b) $x^2 - 2y^4 = 1$.

It is convenient for some problems provided below to use the method of *infinite descent*: Let P be a property such that integers may or may not possess. If an assumption that a positive integer n_0 has property P leads to the existence of a smaller positive integer $n_1 < n_0$ that also satisfies P then no positive integer has that property. This is so because the reasoning that led from n_0 to n_1 , if applied to the latter will produce $n_2 < n_1$ that has property P too. Since the process could be repeated thus leading to an infinitely decreasing sequence of positive integers - which is impossible - the assumption that P is satisfied by a positive integer implies a contradiction and, hence, is false.

Problem 2.6. Prove that the following equations have no solutions in positive integers:

a) $x^4 + y^4 = z^2$; б) $x^4 - 2y^2 = 1$; в) $x^4 + y^2 = z^4$;
 г) $x^4 - 2y^2 = -1$, except $(x, y) = (1, 1)$.

Problem 2.7. Find all the solutions to the following equations in positive integers:

a) $x^2 = 8t^4 - 8t^2 + 1$; б) $x^2 = 2t^4 + 4t^2 + 1$.

Consider the equation $x^4 - 6y^2 = 1$. The minimal solution to Pell's equation $x^2 - 6y^2 = 1$ is $(5, 2)$, so all the solutions (u_k, v_k) are described by

$$u_k + v_k\sqrt{6} = (5 + 2\sqrt{6})^k.$$

Particularly,

$$u_k = \frac{1}{2} \left((5 + 2\sqrt{6})^k + (5 - 2\sqrt{6})^k \right).$$

Note that if equation $x^4 - 6y^2 = 1$ has a positive integer solution then $x^2 = u_k$ for some positive integer k .

Problem 2.8.

- a) Prove that k is even.
- b) Express u_{2l} as a polynomial in u_l .
- c) Prove that k is not divisible by 4.
- d) Prove that $u_{2l+1} - 1$ is a square of an integer.
- e) Find all the solutions to equation $x^4 - 6y^2 = 1$ in positive integers.

Problem 2.9. a) Prove that if equation $x^2 - Dy^2 = 2$ has a solution then it has a single class of solutions. Also, if $\alpha = u + v\sqrt{D}$ generates the fundamental solution to $x^2 - Dy^2 = 2$ then the fundamental solution to $x^2 - Dy^2 = 1$ is given by $\beta = \alpha^2/2$.

b) Prove that equation $x^2 + 1 = 2(t^2 - 1)^2$ has no solutions in positive integers.

c) Prove that if equation $x^2 - Dy^2 = 2$ has an integer solution then equation $x^4 - Dy^2 = 1$ has no solutions in positive integers.

Problem 2.10. Prove that if equation $x^2 - Dy^2 = -2$ has a solution then equation $x^4 - Dy^2 = 1$ has no solutions in positive integers with the sole exception $D = 6$.

3 On the diophantine m -tuples and the equation $x^2 - (k^2 + 1)y^2 = k^2$

Definition. Let n be an integer. A set of m different positive integers is called a *Diophantine m -tuple with the property $D(n)$* or simply *$D(n)$ - m -tuple* if the product of any two of them increased by n is a perfect square.

Problem 3.1. Prove that there does not exist a $D(n)$ -quadruple for any $n \equiv 2 \pmod{4}$.

For the case $n = 1$, the problem of existence of such m -tuples was first studied by Diophantus, and he found a set of four positive rationals with the above property: $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. However the first $D(1)$ -quadruple, the set $\{1, 3, 8, 120\}$, was found by Fermat. The folklore conjecture is that there does not exist a $D(1)$ -quintuple. Recently, Dujella proved that there is no $D(1)$ -sextuple and there are only finitely many $D(1)$ -quintuples.

Problem 3.2. Prove that every $D(1)$ -couple can be extended to a $D(1)$ -triple.

Problem 3.3. Show that every $D(1)$ -triple $\{a, b, c\}$ can be extended to a $D(1)$ -quadruple:

a) Prove that quadratic equation

$$t^2 + a^2 + b^2 + c^2 - 2(ta + tb + tc + ab + bc + ca) - 4abct - 4 = 0$$

has at least one positive integer solution t_0 .

b) Prove that $\{a, b, c, t_0\}$ is a $D(1)$ -quadruple.

In 1969, Baker and Davenport proved that the Fermat set cannot be extended to a $D(1)$ -quintuple.

By definition, if $\{a, b, c\}$ is a $D(1)$ -triple then there exist positive integers u, v, w such that

$$ab + 1 = w^2, \quad bc + 1 = u^2, \quad ca + 1 = v^2.$$

Since

$$bv^2 - au^2 = b(ac + 1) - a(bc + 1) = b - a,$$

the equation $bx^2 - ay^2 = b - a$ has a positive integer solution (v, u) . Notice that Pell's equation $x^2 - aby^2 = 1$ has a solution $(w, 1)$. Let $\alpha = v\sqrt{b} + u\sqrt{a}$ and $\beta = w + \sqrt{ab}$.

Problem 3.4. a) Prove that for any integer l the number $\alpha\beta^l$ is of the form

$$v_l\sqrt{b} + u_l\sqrt{a},$$

where (v_l, u_l) is a solution to equation

$$bx^2 - ay^2 = b - a.$$

b) Prove that if $c \geq |b - a|$ then $u_{-1}, v_{-1} > 0$.

c) Prove that there exists a nonnegative integer c' such that

$$bc' + 1 = u_{-1}^2, \quad ac' + 1 = v_{-1}^2.$$

d) Prove that every $D(1)$ -couple $\{a_1, a_2\}$ has exactly one extension to $D(1)$ -triple $\{a_1, a_2, a_3\}$ such that

$$\max\{a_1, a_2\} < a_3 < 4a_1a_2.$$

Problem 3.5. a) Prove that there are infinitely many arithmetic progressions of length 3 that are also $D(1)$ -triples.

b) Prove that there are no arithmetic progressions of length 4 that are also $D(1)$ -quadruples.

Consider the case $n = -1$. It is easy to find some $D(-1)$ -triples, for example, $\{1, 2, 5\}$, $\{1, 5, 10\}$, $\{1, 26, 37\}$.

Problem 3.6. a) Prove that every $D(-1)$ -couple $\{a_1, a_2\}$ can be extended to a $D(-1)$ -triple $\{a_1, a_2, a_3\}$.

b) Prove that there exists exactly one extension such that

$$\max\{a_1, a_2\} < a_3 \leq 3a_1a_2.$$

It is known that some particular $D(-1)$ -triples, for example, $\{1, 2, 5\}$, cannot be extended to $D(-1)$ -quadruples. Moreover, Dujella and Brown proved that some infinite families of $D(-1)$ -triples cannot be extended to quadruples.

Problem 3.7. Prove that $D(-1)$ -triple $\{n^2 + 1, (n + 1)^2 + 1, (2n + 1)^2 + 4\}$ cannot be extended to a $D(-1)$ -quadruple if

a) $n \equiv 1 \pmod{4}$;

b) $n \equiv 2 \pmod{4}$;

c) $n = 4$.

Let $\{a, b, c\}$ be a $D(-1)$ -triple and let r, s, t be nonnegative integers defined by

$$ab - 1 = r^2, \quad ac - 1 = s^2, \quad bc - 1 = t^2.$$

Assume that there exists a positive integer d that extends this $D(-1)$ -triple to a $D(-1)$ -quadruple. We have

$$ad - 1 = x'^2, \quad bd - 1 = y'^2, \quad cd - 1 = z'^2$$

with nonnegative integers x', y', z' .

Eliminating d from previous equalities, we obtain that x', y', z' form a solution to the following system of equations:

$$ay^2 - bx^2 = b - a,$$

$$az^2 - cx^2 = c - a,$$

$$bz^2 - cy^2 = c - b.$$

Problem 3.8. Suppose $2 \leq a < b < c < d$.

a) Prove that there exist integer solutions (z_0, x_0) and (z_1, y_1) to equations

$$az^2 - cx^2 = c - a \quad \text{and} \quad bz^2 - cy^2 = c - b$$

respectively such that

$$|x_0| < s, \quad 0 < z_0 < c,$$

$$|y_1| < t, \quad 0 < z_1 < c,$$

$$z'\sqrt{a} + x'\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^{2m},$$

$$z'\sqrt{b} + y'\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^{2n}$$

with nonnegative integers m, n .

b) Prove that $z_0 = z_1$.

c) Prove that there exists $d_0 < c$ such that $\{a, b, c, d_0\}$ is also a $D(-1)$ -quadruple.

Suppose there exists at least one $D(-1)$ -quadruple that does not contain number 1. It follows from the previous problem and from the method of infinite descent that there is another $D(-1)$ -quadruple that contains 1.

Existence of $D(-1)$ -quadruples is closely related to the number of classes of solutions to Diophantine equation $x^2 - (k^2 + 1)y^2 = k^2$.

Problem 3.9. a) Prove that for any integer $k > 1$ every class of solutions to the equation $x^2 - (k^2 + 1)y^2 = k^2$ contains exactly one solution x_0, y_0 with $|y_0| < k$.

b) Prove that this equation has at least three different classes of solutions.

Definition. We call a solution (x_0, y_0) to $x^2 - (k^2 + 1)y^2 = k^2$ with $k > 1$ an *exceptional solution* if $x_0 > 0$ and $0 < y_0 < k - 1$.

Problem 3.10. Suppose $\{1, a, b, c\}$ is a $D(-1)$ -quadruple. Prove that there exists $k > 1$ such that equation $x^2 - (k^2 + 1)y^2 = k^2$ has at least two exceptional solutions (or, equivalently, more than 5 classes of solutions).

Problem 3.11. Prove that equation

$$x^2 - (k^2 + 1)y^2 = k^2$$

- a) has no exceptional solutions if $k^2 + 1$ is a prime;
- b) has no exceptional solutions if $k = p^i$ for any odd prime p and positive integer i ;
- c) has at most one exceptional solution if $k = pq$ for any distinct odd primes p, q ;
- d) has at most one exceptional solution if $k = p^i q^j$ for any distinct odd primes p, q and positive integers i, j .

Problem 3.12. Is it possible to find an integer $k > 1$ such that equation

$$x^2 - (k^2 + 1)y^2 = k^2$$

has at least two exceptional solutions?

Acknowledgements.

Authors would like to thank B. Frenkin and M. Surkov for their helpful comments, which improved the exposition.

References.

1. J. E. Shockley, Introduction to Number Theory, Holt, Rinehart and Winston, Inc., New York-Toronto, Ont.-London (1967).

2. M. Waldschmidt, Continued fractions, Ecole de recherche CIMPA-Oujda, Théorie des Nombres et ses Applications, 18 - 29 mai 2015: Oujda (Maroc).
3. JHE Cohn, The Diophantine equation $x^4 - Dy^2 = 1$, Quart. J. Math. Oxford (2) 26 pp. 279-281 (1975).
4. JHE Cohn, Some quartic Diophantine equations, Pacific J. Math. 26(2): 233-243 (1968).
5. A. Dujella, On the size of Diophantine m-tuples. Mathematical Proceedings of the Cambridge Philosophical Society, 132, pp. 23-33 (2002).
6. E. Brown, Sets in which $xy + k$ is always a square. Math. Comp. 45, pp. 613-620 (1985).
7. B. W. Jones, A second variation on a problem of Diophantus and Davenport, Fibonacci Quart. 16 pp. 155-165(1978).
8. A. Dujella, C. Fuchs, Complete solution of a problem of Diophantus and Euler. J. London Math. Soc. 71, pp. 33-52 (2005).
9. K. Matthews, J. Robertson, J. White, On a diophantine equation of Andrej Dujella, Math. Glasnik, 48, number 2, pp. 265-289 (2013).
10. A. Srinivasan, D(-1) quadruples and products of two primes, Glas. Mat. Ser. III 50, pp. 261-268 (2015).
11. M. Le, A. Srinivasan, Note on Dujella's unicity conjecture, Glas. Mat. Vol. 58(78), pp. 59 - 65 (2023).
12. <https://artofproblemsolving.com/community/c6h1186131p5767378>
13. <https://artofproblemsolving.com/community/c6h577945p3408321>
14. <https://artofproblemsolving.com/community/c6h1849278p12466367>
15. <https://artofproblemsolving.com/community/c6h2183565p16353210>
16. <https://artofproblemsolving.com/community/c146h150369p849188>
17. <https://artofproblemsolving.com/community/c146t383f146h150370>
18. <https://artofproblemsolving.com/community/c6h553573p3216004>