# THE QUADRATURE OF THE CIRCLE

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**Part F: Hilbert's third problem**

## † A bit Solid geometry

$\mathbf{F_1}$. Prove that the dihedral angle of a regular tetrahedron is irrational.

$\mathbf{F_2}$. Is it possible to tile space with
a) regular tetrahedra;
b) regular octahedra;
c) regular tetrahedra and regular octahedra together;
d) truncated regular octahedra?

$\mathbf{F_3}$. A regular tetrahedron is allowed to be reflected with respect to any of its faces and to repeat this procedure. Prove that the set of its positions is everywhere dense.

$\mathbf{F_4}$. Suppose from some set of polyhedra (without loss of generality they can be considered convex, by the way, why?) it is possible to assemble both a cube and a regular tetrahedron. We place sticky red paint on the vertices. We assemble a cube, then a tetrahedron, then a cube, then a tetrahedron, and so on. Let the process of multiplying red points stop. Derive a contradiction in this case."

$\mathbf{F_5}$. Let the red paint dry after assembling the cube-tetrahedron-cube-tetrahedron, and it can no longer multiply. Suppose that on each segment with red endpoints and without red interior points, blue points can be placed in such a way that on the adjacent segments to an edge inside the cube or tetrahedron, or on its boundary, the same number of blue points are placed. Derive a contradiction in this case.

$\mathbf{F_6}$. Prove that the desired arrangement of blue points exists.

## † Linear functions

> ➢ Linear
>
> A numerical function $f$ is called *linear* if for any $x, y$ from the domain of definition of $f$, the condition $f(x) + f(y) = f(x + y)$ holds (this condition, in particular, means that $x + y$ also lies in the domain of definition of $f$).

$\mathbf{F_7}$. a) Let $f : \mathbb{Q} \to \mathbb{R}$ be a linear function. Prove that there exists $k \in \mathbb{R}$ such that $f(x) = kx$.
b) Give an example of a linear (numerical) function $f$ that does not have the form $f(x) = kx$ for any $k \in \mathbb{R}$.

- Let's define the concept of the pseudo-volume of a polyhedron $M$ (the Dehn invariant) as the function $f(M) = \sum_i l_i \varphi(\alpha_i)$, where $l_i$ is the length of the $i$-th edge, $\alpha_i$ is the dihedral angle at the $i$-th edge, and $\varphi$ is a numerical function.

$\mathbf{F_8}$. Let the polyhedron $M$ be divided into polyhedrons $M_1$ and $M_2$. What properties must the function $\varphi$ satisfy to ensure that the condition $f(M) = f(M_1) + f(M_2)$ holds?

$\mathbf{F_9}$. a) Prove that if polyhedra are equidecomposable, then their pseudo-volumes are the same.
b) Prove that a cube and a regular tetrahedron are not equidecomposable.
c)* What are the necessary and sufficient conditions for two polyhedra to be equidecomposable?