The quadrature of the circle

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Part E: C.S. problems

- A centrally symmetric convex body (figure) is divided into two congruent sets. Prove that its center lies on their common boundary.

At the same time, the boundary of the parts will be considered as consisting of segments of straight lines and circles (planes and spheres in the spatial case). For polyhedrons, the boundary will be considered as consisting of plane segments.

The expression "figure (body) is divided into parts" can be understood in different ways. Firstly, it can be assumed that the intersection of the parts is empty (i.e., they have no common points).

$\mathbf{E_1}$. In the plane, a set A (possibly non-convex) is given. The set B is the result of some movement of the set A. Can it be that the union of A and B is a centrally symmetric (possibly non-convex) set containing its center of symmetry, while the intersection of the sets A and B is empty?

$\mathbf{E_2}$. In the plane, a set A is given. The set B is the result of some movement of the set A. It turns out that the union of A and B is a flat rectangle (i.e., not a broken line, but a contour together with the interior). Can it be that the center of the rectangle belongs to A but does not belong to B?

- Secondly, intersections of the parts can be allowed along their common boundary (i.e., all common points of the parts lie on their boundaries). Hereafter, we will assume this definition.

$\mathbf{E_3}$. Several points divide a line segment into two congruent sets (the set may consist of several pieces). Prove that one of the points is at its center.

$\mathbf{E_4}$. A circle is divided into 2 parts. Prove that they can be transformed into each other either by a rotation around the center of the circle or by reflection. What can be said about a division into 3 parts?

$\mathbf{E_5}$. Solve the problem **C.S.** for a square.

$\mathbf{E_6}$. Solve the problem **C.S.** for a regular hexagon.

$\mathbf{E_7}$. Solve the problem **C.S.** for an n-dimensional sphere.

$\mathbf{E_8}$.** **Open problem.** Solve the problem **C.S.** for a cube (hypercube).

$\mathbf{E_9}$. In the **C.S.** problem on the plane, separately consider all types of movements by which one part of the division can be transformed into another.

a) Solve the **C.S.** problem if the movement is a translation or a glide reflection.

b)* In the case where a centrally symmetric polygon is divided into two equal parts, transforming into each other by rotation, prove that the center of this rotation lies inside the figure.

c)** **Open problem.** Solve the **C.S.** problem for a polygon if the movement is a rotation.

d)*** **Tough open problem.** Solve the **C.S.** problem for arbitrary centrally symmetric figures in $\mathbb{R}^n$.

$\mathbf{E_{10}}$. The area of a spherical triangle is equal to one-fourth of the area of the sphere. Prove that the sphere can be divided into 4 triangles, each equal to the given one.

$\mathbf{E_{11}}$. a) Cut the plane into equal figures, each bounded by three arcs of circles (a segment is not considered as an arc of a circle).

b)** **Open problem.** Describe all such cuts.

$\mathbf{E_{12}}$. a) Cut a regular hexagon into equal parts such that the center of the hexagon lies inside one of them.

b)** **Open problem.** Is it possible to similarly cut a regular n-gon for $n \neq 3, 4, 6$?