The quadrature of the circle

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. The aim of this project is to investigate the remarkable fact that a square can be transformed into a circle using only parallel shifts.

Part A: Re-partition (a bit of Olympiad material)

A_1. (Squaring the circle 0). Is it possible to cut a square into parts and assemble a circle? (Cuts are segments of straight lines and arcs of circles).

A_2. A convex figure Φ is cut into parts and assembled into a circle (Cuts are segments of straight lines and arcs of circles). Prove that Φ itself is a circle.

A_3. Is it possible to cut two circles into one circle?

Part B: Completeness of invariant systems

➣ Invariant

Let there be a certain process. The *invariant* of this process is a certain quantity (a function of the current state) that does not change regardless of how the process proceeds.

It follows that if the invariant was equal to something at the beginning of the process, it is impossible to bring the system to a state in which this invariant would take on a different value. This is the simple part: to bring the system to a certain state, it is *necessary* that the invariant does not change.

We can pose the reverse question. Suppose there is a certain *invariant* (or several invariants, which we will collectively call a *system of invariants*). Assume that in both state A and state B, this *system of invariants* is the same.

Question: Is it always possible to transition the system from state A to state B?

Answer: It can be either positive or negative. However, if the answer is positive (for any states A and B), then we will call such a *system of invariants* complete.

B_1. a) There are n trees in a circle, with some of them having siskins sitting on them. In one move, one of the siskins flies to the neighbouring tree in a clockwise direction, and another one to the neighbouring tree in a counterclockwise direction. The goal is to gather all the siskins on the first tree (then all the siskins will be happy).

b) Let $n = 43$, with one siskin sitting on each tree. Is it possible to make all the siskins happy?

c) Let $n = 44$, with one siskin sitting on each tree. Is it possible to make all the siskins happy?

d) Find the complete system of invariants for this problem.

e) Let $n = 10$, the siskins sit on every other tree (i.e., on trees with odd numbers). On which trees can all the siskins be gathered?

f) Find the complete system of invariants if the siskins can be gathered on any tree.

B_2. On the island of Serobouromalin, there are gray, brown, and raspberry-coloured chameleons. If two chameleons of different colours meet, they both change their colour to the third colour simultaneously (gray and brown both become raspberry, etc.). The goal is to make all the chameleons the same colour.

a) On the island, there are 13 gray, 15 brown, and 17 raspberry-coloured chameleons. Can it happen that after some time all the chameleons will be of one colour? If yes, what colour will it be?

b) On the island, there are 13 gray, 15 brown, and 16 raspberry-coloured chameleons. Can it happen that after some time all the chameleons will be of one colour? If yes, what color will it be?

c) Find the complete system of invariants for this problem.

B_3. There is a grid rectangle, with some of its cells containing pluses and some containing minuses (there are no empty cells). In one move, it is allowed to change the signs to the opposite in any row or in any column. The goal is to make all the signs pluses.

a) In one corner cell of an 8×8 square there is a minus, and in all other cells there are pluses. Is it possible to make all the signs pluses?

b) In all corner cells of an 8×8 square there are minuses, and in all other cells there are pluses. Is it possible to make all the signs pluses?

c) Find the complete system of invariants for this problem.

B_4. There are n light bulbs on a panel and several buttons, each of which is connected to some of the light bulbs (one light bulb can be connected to several buttons). Pressing a button changes the state of all the light bulbs connected to it to the opposite state. Prove that the number of patterns we can obtain is 2^k for some k.

B_5. We will call an *invariant* such a subset of light bulbs that each button is connected to an even number of light bulbs from this subset. Prove that the number of invariants (including the empty subset) is 2^ℓ for some ℓ. Prove that a certain pattern can be lit if and only if an even number of light bulbs need to be lit in each invariant.

B_6. At the shareholders' meeting of "OAO: if you don't cheat, you won't sell," it was found that for every consecutive 12 months, expenses exceed income, and for every consecutive 11 months, income exceeds expenses. How long can this continue?

B_7. A grid figure is given. The goal is to cover it with L-shaped tiles of three cells (not going beyond the border) in several layers so that each cell of the figure is covered by the same number of cells belonging to the L-shaped tiles.

a) Is it possible to do this for an 8×8 square?

b) Is it possible to do this for a 5×7 rectangle?

c) What sizes of rectangles can be covered this way?

B_8. The grid figure Φ has the following property: for any filling of the cells of an $m \times n$ rectangle with numbers whose sum is positive, it is possible to place the figure Φ in the rectangle so that the sum of the numbers in the cells covered by the figure Φ is positive (the figure Φ can be rotated). Prove that this rectangle can be covered by the figure Φ in several layers.

B_9. There are two figures of equal area. The boundaries of these figures consist of a finite number of line segments and arcs of circles. It is allowed to cut the first figure into a finite number of parts, with the cuts also consisting of line segments and arcs of circles. The goal is to assemble the second figure from these pieces (the pieces can be translated and rotated).

a) Is it possible to make a square from a circle?

b) **First figure:** a circle of radius 2 with a circle of radius 1 cut out of it (the centres of the circles coincide).

Second figure: a sector of a circle with radius 3 and an angle of $\frac{2\pi}{3}$.

Is it possible to make the second figure from the first?

c) A convex figure was transformed into a circle. Prove that it is a circle itself.

d) Is it possible to transform a pair of figures, all of whose boundary segments are arcs of circles, into a convex figure?

e) Find the complete system of invariants for this problem. This will provide an answer to the following question: Given two figures, their boundaries, and possible cuts (line segments and arcs of circles), when can one be transformed into the other?

B_{10}. Given two figures, the boundaries of the figures, and the possible cuts segments of straight lines and arcs of circles. Additionally, homothety is allowed. When can one figure be transformed into the other?

B_{11}. The boundaries of the figures and the possible cuts are a finite union of quadratic segments. When can one figure be transformed into another? It is allowed to perform affine transformations on the pieces.

B_{12}. The boundaries of the figures and the possible cuts are a finite union of quadric segments. When can one figure be transformed into another? It is allowed to perform projective transformations on the pieces.

B_{13}. (Bolyai-Gerwien's Theorem). Prove that any two polygons of equal area on a plane are equidecomposable. Consider spherical and non-Euclidean generalisations.

> ➣ T-equidecomposable
>
> There are two polygons of equal area. We will call them *T-equidecomposable* if the first polygon can be cut into a finite number of parts and the second polygon can be assembled from these parts, with the parts only allowed to be translated (and not rotated).

B_{14}. a) Are a triangle and a square *T-equidecomposable*?
b) Are two equal squares *T-equidecomposable*?
c) Are two parallelograms of equal area *T-equidecomposable*?
d) Find the complete system of invariants for this problem.

B_{15}. Prove that any two parallelepipeds in space of any dimension are *T-equidecomposable*.

B_{16}.* Find the complete system of invariants for *T-equidecomposability* in space of any dimension.

Part C: <u>An arbitrary partitions (homotheties are allowed)</u>

C_1. Homotheties are allowed. The boundaries of the subdivisions can be a union of a countable number of line segments and arcs of circles. Prove that a square can be transformed into a cube.

C_2. Prove that any convex body can be transformed into any other.

Part D: <u>Flows and cells</u>

- This series leads to the main result: the possibility of transforming a square into a circle

Let's start with the preparation (flows on graphs).

D_1. (Hall's Theorem). Let $G = (L, R, E)$ be a bipartite graph. For any k and a k-element subset $L_0 \subseteq L$, the number of vertices in R adjacent to at least one vertex in L_0 is at least k. Then there exists a matching in the graph that covers all vertices in L.

D_2. (König's Theorem). Let G be a bipartite graph. Any matching in G has a size of at most k. Then it is possible to select k vertices such that every edge of the graph has an endpoint among the selected vertices.

D_3. (Menger's Theorem). Between two non-adjacent vertices x and y of a graph, there exist at least t vertex-disjoint paths if and only if, upon the removal of any $t-1$ vertices other than x and y, these two vertices remain in the same connected component.

D_4. The vertices A and B of a graph G are called *equivalent* if there exists a sequence of vertices $A = A_0, A_1, \dots, A_n = B$ such that any two adjacent vertices A_i and A_{i+1} can be connected by k paths without common intermediate vertices. Prove that any two equivalent vertices can be connected by k paths without common edges.

➢ Network

A network is a connected graph in which the capacities of the edges are specified.

- One vertex of the network is considered the *source* (input), and another the *sink* (output).
- In the graph, each edge e is assigned a non-negative number $c(e)$ representing its *capacity*.

➢ Flow

A *flow* in a network G is a pair of functions (f, w), where w is some orientation of all the edges of the network, and $f(e)$ is a function defined on the set of all edges, satisfying the following conditions:

1) $0 \leqslant f(e) \leqslant c(e)$ for all edges of the network;
2) for all vertices v that are not the source or the sink, Kirchhoff's law for electric currents holds:

$$\sum_{e \in \Gamma(v)} f(e) - \sum_{e \in \Gamma'(v)} f(e) = 0,$$

where $\Gamma(v)$ (respectively, $\Gamma'(v)$) is the set of all edges leaving (respectively, entering) v under the orientation w.

- We will denote the flow by f instead of using (f, w).

- *Flow value* is the sum of the values $f(e)$ for all edges e leaving the source (it is easy to see that it is equal to the sum of the values $f(e)$ for all edges e entering the sink).
- *Maximum flow* is a flow whose value is the greatest.
- *Residual network* is a graph G_f whose vertices are the same as in the graph G, and the set of edges is a subset of the edges of the graph G.
  - An edge e belongs to the graph G_f if $c(e) - f(e) > 0$.
- *Augmenting path* is a path $u_1 u_2 \dots u_k$ in the residual network such that u_1 is the source and u_k is the sink.

➢ Cut

A cut is a pair of disjoint sets S and T whose union gives the set of all vertices of the graph G, the source belongs to S, and the sink belongs to T.

- We define $c(S, T)$ as the sum of the capacities $c(e)$ over all edges leading from S to T. The value $f(S, T)$ is defined similarly.

D_5. (Max-Flow Min-Cut Theorem). Let G be a network with capacity $c(e)$ for each edge e, and let f be a flow in this network. The following statements are equivalent:
1. The flow f is maximum.
2. There is no augmenting path in the residual network G_f.
3. There exists a cut (S, T) such that $f(S, T) = c(S, T)$.

D_6. Let the capacity of each edge be an integer. Prove that there is a maximum integer flow in the network (i.e., the flow through each edge is an integer).

$\mathbf{D_7}$. In an $m \times n$ table, each cell contains a number such that the sum of the numbers in each column and the sum of the numbers in each row are integers. Prove that each number can be rounded to some integer (changing each by less than 1) such that the sum in each column and the sum in each row remain unchanged.

➣ Chain and Anti-chain

Let S be a set with a partial order.

- A *chain* is a subset in which any two elements are comparable.
- An *anti-chain* is a subset in which any two elements are incomparable.

$\mathbf{D_8}$. **(Dilworth's Theorem).** Let the set S have no antichain of size greater than k. Then S can be partitioned into k disjoint chains.

- Let's move on to the lattice problems.

$\mathbf{D_9}$. Prove that in the game "Life," (a) a configuration without transformation can be found in the 2010×2010 square; (b) a configuration without transformation can be found on an infinite plane. (The Conway's Game of Life consists of the following rules: in some cells of the grid, there are pieces, while other cells are empty. A piece with fewer than two neighbours dies from loneliness, and a piece with more than three neighbours dies from overpopulation. On an empty cell with three neighbours, a new piece is born. The maximum possible number of neighbours is 8. All events in the cells occur simultaneously for example, every second.)

$\mathbf{D_{10}}$. On some cells of an infinite board there are pieces (at most one per cell), and some cells are empty. A configuration is called *almost full* if there exists a number C such that each piece can be moved a distance not exceeding C (sometimes zero) so that no empty cells remain. A configuration is called *not too empty* if there exists a number D such that the number of empty cells in any square does not exceed DP, where P is the perimeter of the square. Prove that almost full configurations are exactly not too empty.

$\mathbf{D_{11}^*}$. A similar question for n-dimensional space. Let Ω be the set of all grid cubes, divided into smaller cubes. For each cube $L \in \Omega$ and number k, construct the following graph $G(L,k)$. Its vertices are the cells of cube L and two special vertices a and b ("source" and "sink"). The cells of cube L (i.e., the other vertices of $G(L,k)$) are *adjacent* if and only if the distance between them does not exceed k. Furthermore, all free cells in L and only those are adjacent to vertex a, and all cells in L from which a step of length no more than k can be made outside L are adjacent to vertex b.

$\mathbf{D_{12}}$. Prove that under the conditions of problems $\mathbf{D_9}$ and $\mathbf{D_{10}}$, there exists a natural number k such that for any cube $L \in \Omega$ in the graph $G(L,k)$, there exist k vertex-disjoint paths between vertices a and b.

$\mathbf{D_{13}}$. **(Main problem. Lattice version.)** * Let's take a sufficiently large but fixed number k (science can work with $k = 100^{200}$). Consider a grid square of unit area, divided into squares of area $1/n$ each, as $n \to \infty$. Consider a circle of unit area covered by a grid sheet divided into squares with side length $1/n$ (the sides of the squares are parallel to the sides of the original square). Our goal is to select a set of vectors M consisting of k elements and align each square of the square partition with a square covering the circle so that:

1) The corresponding squares differ by a shift of a vector from M.

2) The total area of the squares that do not have a partner is less than δ for sufficiently large n.

Prove that this can be done for any arbitrarily small δ and sufficiently large n.

$\mathbf{D_{14}}$. Prove that there exists a k such that a square and a circle can be almost partitioned into k parts (i.e., for any ε, they can be rearranged to match up to a set of measure ε).

$\mathbf{D_{15}}$*. Prove that there exists a k such that a square and a circle can be partitioned into k parts with exactness up to a set of measure zero.

$\mathbf{D_{16}}$. Generalise the previous points to arbitrary dimensions.

$\mathbf{D_{17}}$. **Research problem.** Try to reduce k.

$\mathbf{D_{18}}$. **Research problem** Try to reduce C in problems $\mathbf{D_{9}}$ and $\mathbf{D_{10}}$.