Distance graphs and Turan's theorem

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By symbol (!) we denote tasks, without solving which it is difficult to move on. The symbol (*) — tasks of increased complexity. The symbol (**) — hard tasks. The further you progress on the project, the more ideas you master. Therefore, tasks (*) and (**) can be skipped and returned to them later using a higher-level technique.

1 To Turan's theorem

The main purpose of this section is to find the minimum possible number of edges in a Turan graph with a given number of vertices and the number of independence. The second goal is to obtain methods of working with Turan graphs that can be used in the rest of the project.

Recall that *independence number* $\alpha(G)$ of a graph G is the size of the maximal set of its vertices that are pairwise not connected with edge.

Theorem 1.1 (Turan, 1941). *Find the minimal possible number of edges in a graph on n vertices with the independence number α.*

1.2. (!) Let a graph have a vertex of degree at least one. Then it can be removed along with the outgoing edges and possibly a few more edges, so that the independence number does not change.

1.3. (!) Let v be a vertex of minimal degree, and N be the union of the set of its neigbours and $\{v\}$.

(a) If you remove all edges with ends in N and make a clique out of N, then the independence number will not increase.
(b) With any method of performing the operation from the previous item, the number of edges in the graph does not increase.
(c) When performing the operation from the first item, the number of edges in the graph does not change if and only if N is both a clique and a component.

A graph is said to be *optimal* if the minimum in Turan's theorem 1.1 achieved on it.

1.4. (!) In an optimal graph
(a) all components are cliques,
(b) sizes of components differ by at most one.

There are many problems in which you need to find a connection between the number of vertices, the minimum possible number of edges and the number of independence in a graph with some restrictions (for example, a graph without triangles, a distance graph). An interesting case is when these constraints prohibit the optimal example from Turan's theorem. Then we will consider the local minimum.

An edge is said to be *excess* if its removing does not change the independence number of the graph.

1.5. Find all excess edges in a (a) path; (b) cycle; (c) clique; (d) square with a diagonal.

A connected graph without excess edges is said to be *turanian*.

1.6. (!) In a turanian graph which is not a clique there are two independent vertices among neighbours of any vertex (in particular, there are no leaves).

1.7. Find the minimal possible number of vertices in a turanian graph with the independence number equal to two.

1.8. Describe all turanian graphs for the numbers of vertices from $\{3, 4, 5, 6\}$.

2 Elementary methods

In this section, we will get acquainted with the basic concepts that will be used in the future to obtain various Turan estimates. Some of these methods will also be useful for obtaining simple linear estimates; the rest of the methods will be useful for more complex techniques, but these methods can be disassembled now to improve understanding of what is happening.

A subset of vertices is said to be *maximal* if the difference between the number of its vertices and the number of edges in the induced subgraph is maximal.

2.1. Descrive all maximal sets of a (a) path; (b) cycle; (c) clique; (d) cycle with a leaf.

A subset of vertices is said to be *independent* if its vertices are pairwise disjoint. I.e. the size of the maximal independent set in G equals $\alpha(G)$.

2.2. (!) The maximal independent set is maximal.

2.3. (!) An edge is excess if and only if there is no maximal set which contains both ends of the edge.

2.4. (!) Suppose that a turanian graph has at leat two vertices. Then for any vertex there are both the maximal independent set containing this vertex and the maximal independent set not containing this vertex.

A subset of vertices is said to be *free* if it lies in the complement of some maximal set. A non-free set is said to be *crucial* if it breaks free when any vertex of it is removed.

2.5. Describe all crucial sets for graphs from Task 2.1.

Problem 2.6. Any vertex of a turanian graph with its neighbours forms a crucial set.

An *articulation point* is a vertex whose removal causes the graph to lose connectivity.

Problem 2.7. Turanian graph does not have articulation points.

Problem 2.8. (a) (*) For a connected graph with independence number α improve the the result of Turan's Theorem 1.1 by $\alpha - 1$. (b) (**) Describe all the cases of equality.

Hint to 2.8.a. Show that an excess edge may be removed. If there are none, remove a vertex of the maximal degree.

Hint to 2.8.b. Suppose, if you remove a vertex and a few extra edges from a turanian graph then it splits into cliques. Then the degrees of all vertices decrease.

Next, we will consider transformations (adding an edge, removing a vertex), during which extra edges appear and the number of components changes. Suppose that as a result of removing a vertex, the graph remains connected, but ceases to be Turanian. Obviously, we can get rid of at least one edge for each increase in the number of components by one. After that, we will show you how to get a lot more!

2.9. (a) For any $n > 2$ show an example of a connected graph on n vertices in which all edges are excess.

(b) When an excess edge is removed then no new excess edges can appear (but can disappear, as may be seen from the previous item).

(c) (*) Which of the five regular polyhedra are turanian graphs? For the rest, come up with a sequence of removing excess edges, leading to the disintegration into several identical turanian parts.

Next, we will use the concept of removing an (ordered) set of extra edges. Specifically, we say that a graph G' is obtained from G by removing a tuple $(e_1, \dots, e_n)$ of excess edges, if there is a tuple $G_0 = G, G_1, \dots, G_{n-1}, G_n = G'$ of graphs such that simultaneously

- sets of vertices for all graphs $G_0, G_1, \dots, G_n$ coinside,
- for any $i \in \{1, 2, \dots, n\}$ the graph G_i is obtained from G_{i-1} by removing the excess edge e_i.

A connected graph is said to be *almost turanian* if its connectivity cannot be broken by removal a few excess edges.

2.10. Give an example of

(a) an almost turanian graph with excess edges;

(b) give an example of a graph with extra edges, depending on the order of removal of which you can get both one turanian graph and two turanian components.

2.11. (a) Let U be a crucial set of a turanian graph. Then after adding a new vertex connected to all vertices from U the resulting graph will remain turanian.

(b) Let U be non-free set of an almost turanian graph. Then after adding a new vertex connected to all vertices from U the resulting graph will remain almost turanian.

3 Simple linear estimates

In this section we will learn how to prove linear bounds on the number of edges in a graph with constraints. Basic actions are removing the vertex of the maximal degree without neighbors, removing the vertex of the minimal degree with neighbors, removing excess edges. In order to have something to improve, we will write out several estimates that are correct for an arbitrary graph.

Here and below we will always denote the number of vertices by n, the number of edges by m, and the independence number by α.

3.1. (!) Suppose that there is a graph with a number n of vertices and the independence number α. Then the number of edges there is at least (a) $n - \alpha$; (b) $2n - 3\alpha$; (c) $3n - 6\alpha$; (d) $4n - 10\alpha$; (e) $5n - 15\alpha$.

The estimates are precise. Proof of any similar linear estimate is a sequence of actions that simplify the graph and allow you to track the change in m, n, α. For example, a vertex whose degree is not less than the coefficient at n can be removed without neighbors (if extra edges appear — so much the better).

If you additionally prohibit some cliques in the graph then the previous bounds can be strengthened.

3.2. (a) (!) Suppose a graph does not have triangles (i.e. cliques on three vertices). Then $m \geqslant 3n - 5\alpha$; (b) The equality $m = 3n - 5\alpha$ is achieved on exactly two connected graphs.

Hint to 3.2.a. It can be assumed that the degrees of vertices in the graph do not exceed 2. It is enough to consider the case of the turanian graph.

Hint to 3.2.b. It can be assumed that in a turanian graph there is a vertex v of degree 3 connected to a vertex of degree 2. What happens if you remove v?

3.3. (a) (!) Suppose that the estimate $m \geqslant 5n - 10\alpha$ holds for all 3-regular graphs without triangles. Prove it for all other graphs without triangles. (b) Set up an infinite series of connected graphs without triangles on which the equality $m = 5n - 10\alpha$ is achieved. (c) (*) Prove the estimate $m \geqslant 5n - 10\alpha$ for 3-regular graphs without triangles.

Hint to 3.3.a. Methods for 3.1.a would be useful. Also, after removal a vertex of the inimal degree d with neighbours, at least d^2 edges are lost. If the graph is not regular then the estimates improves by at least one.

Question: where is used that the graph does not have triangles?

Hint to 3.3.b The simplest example is a cycle on five vertices. It is obtained if you remove a vertex of degree two with neighbours from some graph. To continue the series, adding three vertices at each step, and keep track of the independence number, you will need the concept of a crucial set.

Hint to 3.3.c. By removing a vertex with neighbours, we lose one point. But then we can destroy the graph until we get a point back or reach the end (to which end?)

So, the estimates 3.2 and 3.3 are precise for certain ranges of the ratio $\frac{n}{\alpha} < 3$. At the same time, we do not know the exact estimate for all graphs without triangles (in particular, it would allow us to calculate the Ramsey numbers $R(3, \alpha)$).

Now, after getting acquainted with the technique of removing vertices with neighbors, it is time to get one of the estimates announced in the introduction.

3.4. (!) Let there be a graph on n vertices with the independence number α. Let there be no cliques on four vertices. Then the number of edges in the graph is at least $5n - 12\alpha$.

The task already looks difficult. But the result still needs to be strengthened! To do this, you need to find the points of equality.

3.5. Let in the conditions of 3.4 equality is achieved for some graph. Let d be the minimal degree of a vertex.

(a) Then $2 \leqslant d \leqslant 4$.

(b) What might the set of neighbours of a vertex of degree d look like?

(c) There are exactly three «pairwise non-isomorphic» turanian graphs on which the equality is achieved.

Hint to 3.5. Prove that the equality case is the only one, first for $d = 2$, then for $d = 3$, then for $d = 4$. In a 4-regular graph, it is convenient to cling to a cycle on four vertices with a diagonal.

It seems that in order to strengthen the estimation for a distance graph, it is necessary to use the non-distanceness of a large list of subgraphs, an bruteforce comes out. Thus, the estimate for the number m of edges can be strengthened to $m \geqslant 8\frac{1}{3}\alpha$ (if you prohibit the non-distance example from 3.5) or even $m \geqslant 8\frac{2}{3}\alpha$ (if you prohibit a pentagonal pyramid without one side edge and a pair of vertices with by three common neighbors) at $n = 4\alpha$. It is not enough! We will choose another way. For an arbitrary turanian graph, we obtain estimates (a) $m \geqslant 3n - 4\alpha - 2$; (b) $m \geqslant 4n - 7\alpha - 3$; (c) $m \geqslant 5n - 11\alpha - 4$.

There are many cases of equality (in particular, $m = 3n - 4\alpha - 2$ in the examples from 3.2.b. The estimate $m \geqslant 4n - 7\alpha - 3$ for graphs without cliques on four vertices is precise when $n \leqslant 4\alpha - 1$. However, at $n \geqslant 4\alpha$, it can be further enhanced. Let there be a turanian graph on n vertices with the independence number α. Let there be no cliques on four vertices. Then the number of edges there is at least (a) $5n - 11\alpha - 2$; (b) $6n - 15\alpha - 1$, except for the graph of the quadrangular antiprism.

Connectivity and the absence of excess edges make it possible to add a free term to the estimate. Thus, the cases of equality for $\alpha = 2$ are preserved, and with large α the estimate improves. In addition, inconvenient equality cases can be bypassed. The usual vertex removal technique does not allow us to prove such estimates, because after removing excess edges the graph can crumble into thousands of fragments, and the free terms for different parts add up! Therefore, we will devote the next two chapters to a technique that allows us to solve these problems.

4 Splits

In this section we will get acquainted with the concept of splits, which will help us transform graphs so that it is easier for them to obtain Turan estimates. The idea is to add an edge to the turanian graph or remove a vertex, making several other edges excess. There are two problems along the way. First, the added edge may be part of a forbidden subgraph. To solve this problem, we will color such edges red and remove all restrictions from them. We will also need red vertices. Note that any vertex of a red edge can easily and irreversibly turn into a red vertex. We will call the usual vertices and edges blue.

Secondly, removing unnecessary edges can lead to an increase in the number of components. A vertex v of a turanian graph is said to be *unstable* if, when removing the vertex v and several excess edges, the graph can fall apart. A pair (u, v) of independent vertices of a turanian graph is said to be *special* if, when adding an edge uv and removing several excess edges, the graph can fall apart.

4.1. (a) (!) Any vertex of a special pair is unstable. (b) (!) Any pair of vertices of an odd cycle is special. (c) Give an example of a turanian graph without unstable vertices, which is not the clique.

As you may see from 4.1.b, there may be many parts and a few excess edges. That is why we do not want to leave the obtained graph in such a form. So, the following theorem holds.

Problem 4.2 (**). (a) **A split over edge**. Suppose that when adding a red edge e and removing several excess edges, the graph splits into almost turanian components $G_1, \dots, G_t$, and e lies in G_1. Then one can add one red vertex to the components $G_2, \dots, G_t$, while the independence number will not change, and the total number of edges in the graph will increase by at most $t - 1$.

(b) **A split over vertex**. Suppose that when an unstable vertex v and a few excess edges are removed, the graph splits into almost turanian components $G_1, \dots, G_t$. Then one can add one red vertex to all components $G_1, \dots, G_t$, while the independence number will not change, and the total number of edges in the graph will increase by at most $t - 1$.

Solving Problem 4.2 within the framework of the methods of this part is not so simple (including since the result of the split and even the number of parts are not uniquely determined). This problem will be solved further using hypergraphs. In the future, you can use the result of Problem 4.2 without proof (except for problems that are special cases of this theorem).

4.3. (a) (!) Split a pentagon over a vertex.

(b) (!) Split an odd cycle over a vertex.

(c) (!) Split (over a vertex and over an edge) a turanian graph on $n > 3$ vertices with a pair of adjacent vertices of degrees equal to 2.

(d) Which method from item (b) guarantees a split exactly on two parts?

(e) Give an example of a connected graph with excess edges, containing an unstable vertex, that cannot be split (too many edges will need to be added).

This section is devoted to the split into two parts (let's call such a split *simple*). Our goal is not only to prove the splitting theorem, but also to find points of equality (let's call a split into t parts, in which the number of edges in the graph increases by exactly $t-1$, *charge-free*).

Let G be a turanian graph. Let e be an edge from a complement of G. A graph is said to be *e-presplitted*, if it (1) is a union of two almost turanian components G_1, G_2, (2) is obtained from G by adding e and removing all edges that connect G_1 to G_2, (3) has the same independence number as G.

4.4. Find, if possible, a e-presplitted graph for (a) (!) a cycle on five vertices, (b) the Moser spindle.

A simple split over edge.

Let G be a turanian graph on m edges. A graph is obtained by *a simple split over an edge* e if the number of edges in it is at most $m+1$, and it is obtained by adding a (red) vertex in the component of some e-presplitted graph that does not contains e.

4.5. (!) If there is a e-presplitted graph for a turanian graph then a simple split is possible.

Hint to 4.5. Use the fact that edges connecting G_1 with G_2 were not excess initially. Then find a crucial set in G_2.

A *cost* of a graph without excess edges is the difference between the numbers of edges and of components. When the cost is reduced by one, saving the difference between the numbers of vertices and of components, we obtain a coin.

Problem 4.6 (**A simple charge-free split over edge**)**.** Let it be possible to split the graph G along the edge e into almost turanian parts G_1, G_2. Suppose that the number of edges in the graph increases with this split (in other words, we cannot get a coin with this split). Let e lie in G_1. Then

(a) in G from each vertex of G_1 leads at most one edge to the vertices of G_2;

(b) the vertices of G_2, from which in G lead edges to the vertices of G_1, in union with the ends of the edge e form an independent set;

(c) the graph G can be split by any edge constricting the vertices of the independent set from the previous item.

Problem 4.7 (**An elementary split**)**.** Let a turanian graph G consists of two non-empty subgraphs G_1, G_2, and two vertices u and v, with no edges leading from the subgraph G_1 to G_2. Then

(a) the vertices u and v are independent;

(b) after the edge uv is added, all edges leading from u and v to one of the subgraphs (let G_2) will become excess;

(c) the vertices of the graph G_2 connected to u or v form a crucial set U;

(d) after adding a new red vertex x connected to the vertices of the set U, we obtain a turanian graph $G_2 \cup \{x\}$;

(e) the graph G_1 with drawn edge uv is also turanian.

Problem 4.8. Let there are two turanian graphs with at least three vertices in each of them. How to «assemble from them» one turanian graph by reducing the numbers of vertices and of edges by one?

Hint. It is important to prove that there are no excess edges in the obtained graph. Problem 4.8 allows us to convolve an optimal graph into a turanian graph in a few steps (in a colossal number of ways). We call such an action recombination of graphs by an edge of one graph and a vertex of another. Our goal is to prove that the constructed example will have the minimum number of edges (among all turanian graphs with the same characteristics).

Problem 4.9. (!) Perform an elementary split in the following turanian graphs different from clique and containing:
(a) a vertex of degree 2;
(b) a vertex of degree 3 entering two triangles.

Below we use the following notation (let the subgraph G_1 is above and G_2 is below): X is a articulation point, XX is a possibility of an elementary split.

4.10. (!) Let a turanian graph G consist of a vertex and subgraphs G_1, G_2, and there is exactly one edge connecting G_1 to G_2 (case XI).
(a) Specify which elementary splits over an edge are possible.
(b) How to get a red edge in each part?

4.11. (!) Suppose a turanian graph G consists of subgraphs G_1, G_2, and there are exactly two edges connecting G_1 to G_2 (case II). What is obtained after two elementary splits?

Now let's go back to Tasks 3.4 and 3.5. If you have not solved all the items, apply the split technique. Consider that the red vertex is worth 4 points (but do not rush to remove the red vertex of degree 4 from the graph).

Problem 4.12. (a) In Task 3.5.b, the «worst cases» of the position of a vertex of degree 3 are obtained. What are the possible splits in these cases?
(b) How many points can be guaranteed in these cases when $\alpha > 2$? Construct an example of a turanian graph with $\alpha = 3$ containing a vertex of degree 3 worth 1 point (equality $m = 5n - 12\alpha + 1$).
(c) (*) Show that for $\alpha > 3$ the turanian graph containing a vertex of degree 3 is worth more than one point (inequality $m \geqslant 5n - 12\alpha + 2$).

Problem 4.13. Let a turanian graph contains a triangle of vertices of degree 3 that are not included in other triangles.
(a) Which edges can be used to split a graph over?
(b) Give an example of such a graph with 8 vertices. What is obtained by splitting?
(c) (*) Let some edges of the graph contain more than one vertex. (The graph becomes a hypergraph, which can contain both simple edges and hyperedges). Split the graph from (a,b) over a hyperedge. Split the hypergraph obtained from (b) repeatedly.
(d) (*) Give a general definition of the elementary split of a turanian hypergraph. What conditions on the maximal independent set in G_1 or G_2 are sufficient to allow an elementary split?

A simple split over vertex.

Let G be a turanian graph on m edges, and v is its unstable vertex. A graph is said to be *v-predsplitted* if it: (1) is two almost turanian components G_1, G_2, (2) is obtained from G by removing v and all edges connecting G_1 to G_2, (3) has the same independence number α as G. A graph is obtained by *simple split of graph G over the vertex v* if the number of edges in it is at most $m + 1$, and it is obtained by adding one vertex to each of both components of some v-presplitted graph.

4.14. (!) If there exists a v-presplitted graph then a simple split is possible.

4.15. (*) Suppose a turanian graph G consists of two vertices and subgraphs G_1, G_2, and there is exactly one edge connecting G_1 to G_2 (case XXI). Prove that the graph can be split over one of the vertices.

Problem 4.16 (A symmetric split over vertex). Suppose that in the previous problem it is impossible to split the graph over a vertex v so that to obtain a coin. Moreover, let it be impossible to split the graph over an edge incident to v. Consider any v-presplitted graph $G_1 \sqcup G_2$, where G_1, G_2 are its almost turanian components.

(a) In G, each vertex of graph G_1 has at most one edge leading from it to the vertices of graph G_2.

(b) The vertices of G_1, from which in G lead edges to the vertices of G_2, in union with $\{v\}$ form an independent set U.

(c) $\alpha(G_i) \geqslant 3$ for $i = 1, 2$.

(d) (*) Give an example of a turanian graph for which a symmetric split over one of the vertices is possible.

So, we know a lot about simple splits and almost everything about simple charge-free splits, we believe in the general case of the splitting theorem.

The first application of this technique will be linear estimates with free term for turanian graphs.

4.17. For a Turanian graph on n vertices with independence number α prove the following lower bounds on the number of edges: (a) (!) $3n - 4\alpha - 2$; (b) $4n - 7\alpha - 3$; (c) (*) $kn - \frac{k^2-k+2}{2}\alpha - k + 1$ for any integer $k \geqslant 3$.

To apply this technique, go back to Task 3.3 (if you have not solved all the items yet, go back to them armed with the splits and inequalities technique 4.12). Consider that a red vertex is worth 3 points, but do not be in a hurry to remove red vertices of degree 3 — rather split the graph.

Problem 4.18. (*) Let a turanian graph does not contain triangles. Consider the case of the equality $m = 5n - 10\alpha$.

(a) The graph is not 3-regular.

(b) Let one of the vertices be split. Then the splits follow one after another until the graph is disintegrated into cliques.

(c) All splits are simple charge-free splits. If the graph can be split, the equality $n = 3\alpha - 1$ holds.

(d) All splits are elementary.

(e) For every $\alpha > 1$ there is a single example with a possible split. All vertices in the example are unstable.

(f) Are there any other examples?

5 A removal of a vertex with neighbours

So, we have learned how to correctly remove any vertex of a turanian graph without neighbours. Now we want to learn how to remove a vertex with neighbours.

A vertex of the smallest degree in a turanian graph is said to be *standard* if removing this vertex and all its neighbours results in a almost turanian graph. It seems that we had no problems with removing the standard vertex. But in fact, it is often possible to improve the result. For example, for a vertex of degree 2 we want the removal to coincide with a split.

Let there be a standard vertex v of a turanian graph that is not clique. Let us call a *removal of the vertex v with neighbours* the following process:

1) two non-adjacent neighbours u_1, u_2 of v are selected;
2) all neighbours of v except u_1 and u_2 are removed, and then all excess edges;
3) the resulting graph is split over the edge u_1u_2;
4) the removed vertices are returned to the clique vu_1u_2.

5.1. Если возможно, удалите с соседями: If possible, remove with neighbours
1) (!) a vertex of the pentagon;
2) a vertex of the quadrangular antiprism;
3) (*) a vertex of the icosahedron.

Problem 5.2. The removal of a standard vertex v with neighbours is defined correctly, i.e.
(a) v has to non-adjacent neighbours u_1, u_2;
(b) at the second step the removal of vertices and excess edges cannot lead to the violation of graph connectivity;
(c) at the third step the graph can be split along edge u_1u_2, and the split is simple.

Problem 5.3. A removal of a standard vertex with neighbours is a split (i.e. the number of edges is increased by at most one).

Nevertheless, the number of red objects increases by $d - k + 2$, where d is the degree of the vertex v to be removed, and k is the minimum size of the forbidden clique (provided that v, its neighbours and the edges leaving them are all blue, otherwise it is better). For example, in a four-vertex clique-free graph, removing a vertex of degree 3 with neighbours leads to the same result as a simple split. As with splits, we are interested in cases where removing a vertex does not give us any coins (in the sense of splits).

Problem 5.4. (*) Let it be impossible to obtain a coin when removing a vertex v of degree d with neighbours. Then one of two alternatives is fulfilled:
1) the vertex v lies in a clique on d vertices, and the graph can be split over edge;
2) the vertex v has independent neighbours u_1, u_2, u_3;
all other neighbours of v are connected only to v, u_1, u_2, u_3 and among themselves;
all neighbours of v have degree d;
if $d > 3$ and the vertex u_i is standard, then removing u_i with its neighbours yields a coin.

In the second case, at $d > 3$ there appear several pairwise connected vertices having the same set of neighbours. We will call such vertices *clones*.

Problem 5.5. (!) Let a vertex v is not a standard vertex. Then the process of removing this vertex with its neighbours is impossible; however, it can be easily modified.
(a) What to do if at some point of the second step the graph ceases to be almost turanian?
(b) What if the degree of any of the neighbours of a vertex v is less than the degree of v itself? Show that we can not only continue the process, but also get a coin for each of the vertices that we have removed by this point.

Problem 5.6. Using the previous problem, prove that the example constructed in Problem 4.8 is optimal.

Thus, we have obtained an exact lower bound for the number of edges in a turanian graph. Next, we will come closer to the description of optimal graphs.

For a turanian graph G, let us call the *summarised* graph G' for which the following properties hold:
1) graphs G, G' have the same independence number α;
2) the number of vertices in G' is by $\alpha - 1$ greater than the number of vertices in G;
3) components of the graph G' are cliques whose sizes differ by at most one.

5.7. (a) The summarised graph is uniquely constructed by the number of vertices and the independence number of the original graph.
(b) The number of edges of the original graph is not more than by $\alpha - 1$ less than the number of edges of the summerised graph.

5.8. (a) The degree of any vertex of the summmarised graph is at least two.
(b) The average degree of the vertices of a turanian graph is not less than the average degree of the vertices of its summmarised graph.
(c) Equalities in the previous points are achieved only for odd cycles.
(d) The degree of some vertex of the optimal turanian graph is not greater than the degree of some vertex of the summmarised graph.

Now we obtain additional conditions on the graphs for which we achieve equality (i.e. we describe optimal turanian graphs).

Problem 5.9. (a) The degree of any vertex of the optimal turanian graph is not less than the smallest degree of the vertices of the summarised graph.
(b) Let the optimal turanian graph cannot be split over a vertex v. Then the degree of v is not greater than the largest degree of vertices of the summarised graph.

In the previous problem the following notion practically appeared. Let us call *stable* a turanian graph which is not clique and which cannot be split over a vertex.

Problem 5.10. (a) Give an example of a stable graph.
(b) * Are there any optimal stable graphs?
(c) The degree of any vertex of the optimal stable graph is equal to the greatest degree of the vertices of its final graph.