INVARIANTS OF ALMOST EMBEDDINGS OF GRAPHS IN THE PLANE: RESULTS AND PROBLEMS

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Abstract. A graph drawing in the plane is called an *almost embedding* if images of any two non-adjacent simplices (i.e. vertices or edges) are disjoint. We introduce integer invariants of almost embeddings: winding number, cyclic and triodic Wu numbers. We construct almost embeddings realizing some values of these invariants. We prove some relations between the invariants. We study values realizable as invariants of some almost embedding, but not of any embedding.

This paper is expository and is accessible to mathematicians not specialized in the area (and to students). However elementary, this paper is motivated by frontline of research.

Contents

A classical subject is study of planar graph drawings without self-intersections (i.e. embeddings or plane graphs). It is also interesting to study graph drawings having 'moderate' self-intersections, e.g. almost embeddings (see definition near Figures 4.1 and 4.2).

For relation to modern research see remark at the end of §5, papers [Sk18, KS20, IKN+, Ga23, Bo] and the references therein. We do not know proofs of Conjectures 5.4.b, 5.7.b, 6.4.b, 6.5.b, 7.1.b, or solution of Problem 5.8.

In this text we expose a theory as a sequence of problems, see e.g. [HC19], [Sk21m, Introduction, Learning by doing problems] and the references therein. Most problems are useful theoretical facts. So this text could in principle be read even without solving problems or looking to §8. Problems are numbered, the words 'problem' are omitted. If a mathematical statement is formulated as a problem, then the objective is to prove this statement. Open-ended questions are called **riddles**; here one must come up with a clear wording, and a proof. *If a problem is named 'theorem' ('lemma', 'corollary', etc.), then this statement is considered to be more important.* Usually we *formulate* beautiful or important statement *before* giving a sequence of results (lemmas, assertions, etc.) which constitute its *proof*. We give hints on that after the statements but we do not want to deprive you of the pleasure of finding the right moment when you finally are ready to prove the statement. Important definitions are highlighted in **bold** for easy navigation.

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1. Winding number: definition and discussion

In the plane let $O, A, B, A_1, \dots, A_m$ be points.

Assume that $A \neq O$ and $B \neq O$ (but possibly $A = B$). Recall that the *oriented (a.k.a. directed) angle* $\angle AOB$ is the number $t \in (-\pi, \pi]$ such that the vector $\overrightarrow{OB}$ is codirected to the vector obtained from $\overrightarrow{OA}$ by the rotation through t. (If you are familiar with complex numbers, you can regard vectors in the plane as complex numbers, and rewrite this condition as $\overrightarrow{OB} \upuparrows e^{it}\overrightarrow{OA}$.)

A **polygonal line** $A_1 \dots A_m$ is the (ordered) set $(\,A_1A_2,\ A_2A_3,\ \dots,\ A_{m-1}A_m\,)$ of segments. A **closed polygonal line** $A_1 \dots A_m$ is the set $(\,A_1A_2,\ A_2A_3,\ \dots,\ A_{m-1}A_m,\ A_mA_1\,)$ of segments.[1]

Let $A_1 \dots A_m$ be a closed polygonal line not passing through O. The **winding number** $w(A_1 \dots A_m, O)$ of $A_1 \dots A_m$ around O is the number of revolutions during the rotation of vector whose origin is O, and whose endpoint goes along the polygonal line in positive direction. Rigorously,

$$2\pi \cdot w(A_1 \dots A_m, O) := \angle A_1OA_2 + \angle A_2OA_3 + \ldots + \angle A_{m-1}OA_m + \angle A_mOA_1$$

is the sum of the oriented angles.

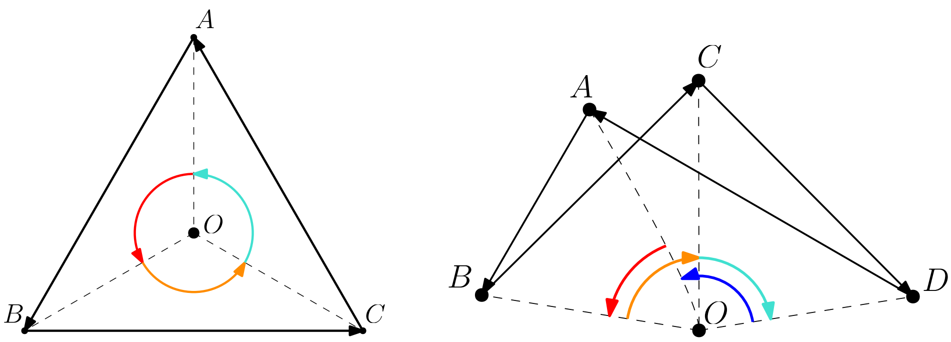

Figure 1.1. $w(ABC, O) = +1$ and $w(ABCD, O) = 0$

E.g. in Figure 1.1

$$w(ABC, O) = \frac{1}{2\pi}(\angle AOB + \angle BOC + \angle COA) = +1 \quad \text{and}$$

$$w(ABCD, O) = \frac{1}{2\pi}(\angle AOB + \angle BOC + \angle COD + \angle DOA) = \frac{1}{2\pi}(\angle BOD + \angle DOB) = 0.$$

1.1. The winding number w is an integer.

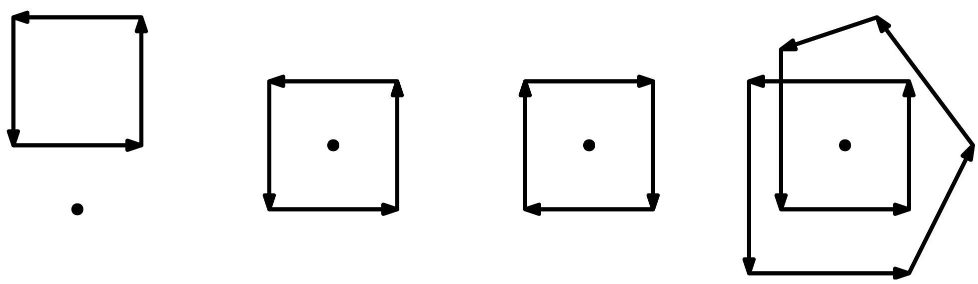

Figure 1.2. The winding numbers equal 0, +1, −1, +2

[1]The set of segments is not the same as the union of segments. Thus, strictly speaking, the polygonal line (defined here) is not a subset of the plane. So 'oriented' or 'non-oriented' is not formally applicable to polygonal lines. Still, we sometimes work with the set of segments as with their union, e.g. we write 'a polygonal line not passing through a point'. The notion of polygonal line defined here is close to what is sometimes understood as 'oriented polygonal line'.

1.2. (a) The winding number of (the outline of) any convex polygon around any point in its exterior (respectively interior) is 0 (respectively ± 1). See Figure 1.2.

(b) Let ABC be a regular triangle and O its center. Find $w(ABCABC, O)$.

(c) For any integer n and any point O in the plane there is a closed polygonal line whose winding number around O is n.

(d) Give an example of a closed polygonal line L in the plane such that $w(L, O) = 0$ for any point $O \in \mathbb{R}^2 - L$.

The analogue of Assertion 1.2.a is correct for any closed polygonal line without self-intersections. (Depending on the exposition, this is either a corollary of the *Jordan Curve Theorem*, or a lemma in its proof.) The result of Problem 1.2.b shows that winding numbers for distinct polygonal lines with the same union of their segments can be distinct.

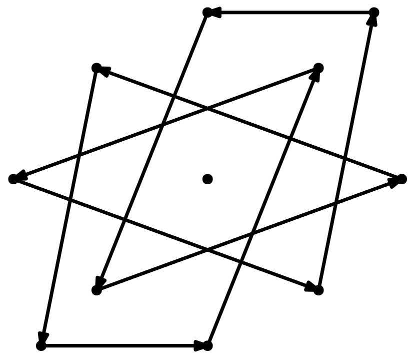

FIGURE 1.3. A closed polygonal line symmetric w.r.t. a point; the winding number equals 3

Theorem 1.3 (Borsuk-Ulam). *Suppose that a closed polygonal line $A_1 \dots A_{2k}$ does not pass through a point O, and is symmetric w.r.t. O (i.e. O is the midpoint of the segment A_jA_{k+j} for every $j = 1, \dots, k$). Then the winding number is odd.*

The following notion and results would be helpful.

Let $A_1 \dots A_m$ be a polygonal line not passing through a point O. Define the real number $w'(A_1 \dots A_m, O)$ by

$$2\pi \cdot w'(A_1 \dots A_m, O) := \angle A_1OA_2 + \angle A_2OA_3 + \ldots + \angle A_{m-1}OA_m.$$

Clearly,

- $2\pi w(A_1 \dots A_m, O) = 2\pi w'(A_1 \dots A_m, O) + \angle A_mOA_1$;
- if points $A_2, \dots, A_{m-1}$ lie in the interior of the angle $\angle A_1OA_m$, then $w'(A_1 \dots A_m, O) = \angle A_1OA_m$.

1.4. (a) We have $\angle A_1OA_m = 2\pi w'(A_1 \dots A_m, O) + 2\pi k$ for some integer k.

(b) We have $w(A_1 \dots A_m, O) = w'(A_1 \dots A_j, O) + w'(A_j \dots A_mA_1, O)$ for every $j = 1, \dots, m$.

Denote by $\overline{l}$ the polygonal line obtained from a polygonal line l by passing in the reverse direction.

1.5. In the plane let O, A, B be three pairwise distinct points.

(a) Let l_1, l_2, l_3 polygonal lines joining A to B, and not passing through O. Then

$$w(l_1\overline{l_2}, O) + w\left(l_2\overline{l_3}, O\right) = w(l_1\overline{l_3}, O).$$

(b) For any three integers n_1, n_2, n_3 such that $n_1 + n_2 = n_3$ there are three polygonal lines l_1, l_2, l_3 joining A to B, not passing through O, and such that

$$w(l_1\overline{l_2}, O) = n_1, \qquad w\left(l_2\overline{l_3}, O\right) = n_2 \quad \text{and} \quad w(l_1\overline{l_3}, O) = n_3.$$

In Assertion 2.1.a, Problems 1.5.b, 2.1.bc, and other *examples* in this text (as opposed to *assertions* distinct from 2.1.a) you may give an heuristic rather than rigorous proof, unless you or your advisor realize that this leads to a confusion.

1.6. Let $A_1A_2A_3$ be a regular triangle, and O its center. For $m = 1, 2, 3$ let l_m be a polygonal line disjoint with the ray OA_m, and joining A_{m+1} to A_{m+2}, where the numbering is modulo 3. Then $w(l_1l_2l_3, O) = \pm 1$.

2. Winding number and intersections

This section is only used in sketches of proofs of Theorems 5.2 and 5.5.a in §5.

2.1. (a) In the plane points P_0, P_1 are joined by a polygonal line disjoint from a closed polygonal line L. Then $w(L, P_0) = w(L, P_1)$.

Hint: use *considerations of continuity.*

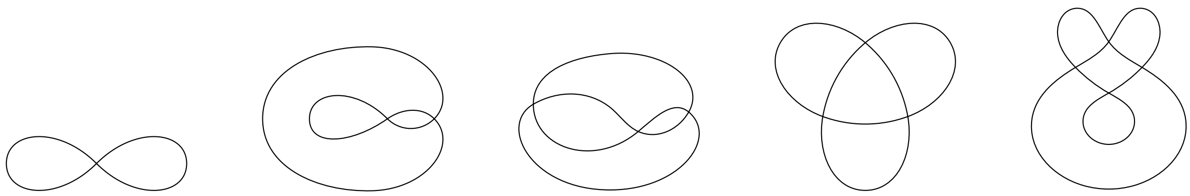

Figure 2.1. Some closed polygonal lines

(b) Take closed polygonal lines L in the plane shown in Figure 2.1 (with some orientations; any of the polygonal lines does not go twice through any segment, and does not significantly change its direction at any point). Color the complement $\mathbb{R}^2 - L$ by parity of the winding number of L.

(c) For closed polygonal lines L from (b) color the complement $\mathbb{R}^2 - L$ by the winding number of L.

2.2. Take a closed and a non-closed polygonal lines L and P in the plane, all whose vertices are pairwise distinct and are in general position. Let P_0 and P_1 be the starting point and the endpoint of P. Assume that $P_0, P_1 \notin L$.

(a) Then $|L \cap P| \equiv w(P_1, L) - w(P_0, L) \mod 2$. (This is a discrete version of the *Stokes theorem.*)

Hint. It suffices to prove this fact for a sufficiently small segment P' such that $P' \subset P$.

(b) If P_1 is far away from L, then $w(P_0, L) \equiv |L \cap P| \mod 2$, and $w(P_0, L)$ equals to the sum of signs of intersection points of P and L (defined in [Sk18, §1.3]).

Let L be a closed polygonal line in the plane, all whose vertices are pairwise distinct and are in general position. By Assertion 2.2.a the complement to L has a 'chess-board' coloring, i.e. a coloring such that the adjacent domains have different colors. The *modulo two interior* of L is the union of black domains for a chess-board coloring (provided the infinite domain is white). In other words, this is the set of all points $x \in \mathbb{R}^2 - L$ for which there is a polygonal line P

- joining x to a point 'far away' from L (i.e. outside the convex hill of L),
- intersecting L at an odd number of points, and
- such that all the vertices of L and P are pairwise distinct, and are in general position.

This is well-defined by [Sk18, Parity Lemma 1.3.3].

3. Winding numbers of graph drawings

Remark (some rigorous definitions). You can work with the notions defined here at an intuitive level, before you or your advisor realize that this leads to a confusion.

A (finite) **graph** (V, E) is a finite set V together with a collection $E \subset \binom{V}{2}$ of two-element subsets of V (i.e., of non-ordered pairs of distinct elements). (The common term for this notion is *a graph without loops and multiple edges* or *a simple graph.*) The elements of this finite set V are called *vertices*. The pairs of vertices from E are called *edges*. The edge joining vertices i and j is denoted by ij (not by (i,j) to avoid confusion with ordered pairs). A cycle in a graph is denoted by listing its vertices in their order (without commas).

Informally speaking, a graph is planar if it can be drawn 'without self-intersections' in the plane. Rigorously, a graph (V,E) is called **planar** (or piecewise-linearly embeddable into the plane) if in the plane there exist

- a set of $|V|$ points corresponding to the vertices, and
- a set of non-self-intersecting polygonal lines joining pairs (of points) from E

such that none of the polygonal lines intersects the interior of any other polygonal line.[2]

Denote by

- $[n]$ the set $\{1,2,\dots,n\}$;
- K_n the complete graph with the vertex set $[n]$;
- $K_{m,n}$ the complete bipartite graph with parts $[m]$ and $[n]'$ (we denote by A' a copy of A).

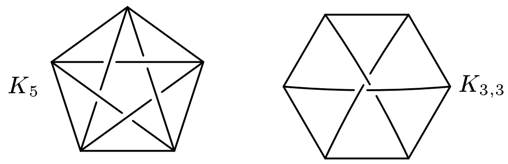

FIGURE 3.1. Non-planar graphs K_5 and $K_{3,3}$

We consider graph drawings in the plane such that the edges are drawn as polygonal lines, and intersections are allowed. Let us give rigorous definitions. Let K be a graph with V vertices. A (piecewise-linear) **map** $f: K \to \mathbb{R}^2$ of K to the plane is

- a collection of V points in the plane corresponding to the vertices, and
- a collection of (non-closed) polygonal lines in the plane joining those pairs of points from the collection which correspond to pairs of adjacent vertices.[3]

The **restriction** $f|_\sigma$ to an edge σ is the corresponding polygonal line. The **image** $f(\sigma)$ of edge σ is the union of edges of $f|_\sigma$. The **image** of a collection of edges is the union of images of all the edges from the collection.

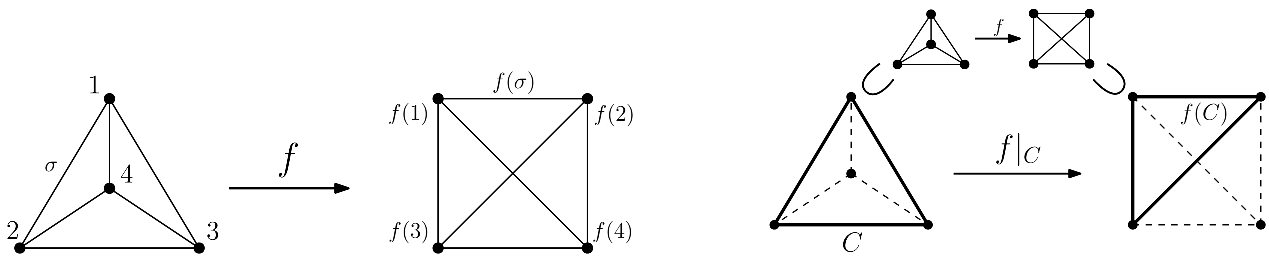

FIGURE 3.2. A map $f: K_4 \to \mathbb{R}^2$ (left) and its restriction $f|_C$ (right)

Let $C = v_1 \dots v_n$ be a directed (i.e. oriented) cycle in K. E.g. for $j = 1,2,3,4$ denote by C_j the directed cycle in K_4 obtained by deleting j from 1234. Let $f: K \to \mathbb{R}^2$ be a map. The **restriction** $f|_C : C \to \mathbb{R}^2$ of f is the closed polygonal line 'formed' by the polygonal lines $f|_{v_1v_2}, \dots, f|_{v_{n-1}v_n}, f|_{v_nv_1}$ in this order.

[2]Then any two of the polygonal lines either are disjoint or intersect by a common end vertex. We do not require that 'no isolated vertex lies on any of the polygonal lines' because this property can always be achieved.

[3]This is a slight abuse of terminology. A polygonal line has a starting point and an endpoint, so this is a definition of a map of an oriented graph. Two maps from oriented graphs are *equivalent* if one of them is obtained from the other by change of orientations of some edges, and one of the corresponding collections of polygonal lines is obtained from the other by passing the corresponding polygonal lines in the reverse order. Rigorously speaking, a map of a graph is an equivalence class under such an equivalence relation.

3.1. For any map $f : K_4 \to \mathbb{R}^2$ and point $O \in \mathbb{R}^2 - f(K_4)$ we have $\sum_{j=1}^4 (-1)^j w(f|_{C_j}, O) = 0$:

$$-w(f|_{234}, O) + w(f|_{134}, O) - w(f|_{124}, O) + w(f|_{123}, O) = 0.$$

For a vertex v in K such that $f(v) \notin f(C)$ denote

$$w_f(C, v) := w(f|_C, f(v)).$$

3.2. (a) For any integer n there is a map $f : K_4 \to \mathbb{R}^2$ such that

$$f(1) \notin f(C_1), \quad w_f(C_1, 1) = n, \quad \text{and} \quad f(j) \notin f(C_j), \quad w_f(C_j, j) = 0 \quad \text{for every} \quad j = 2, 3, 4.$$

(b) For any integers n_1, n_2, n_3, n_4 there is a map $f : K_4 \to \mathbb{R}^2$ such that $f(j) \notin f(C_j)$ and $w_f(C_j, j) = n_j$ for every $j = 1, 2, 3, 4$.

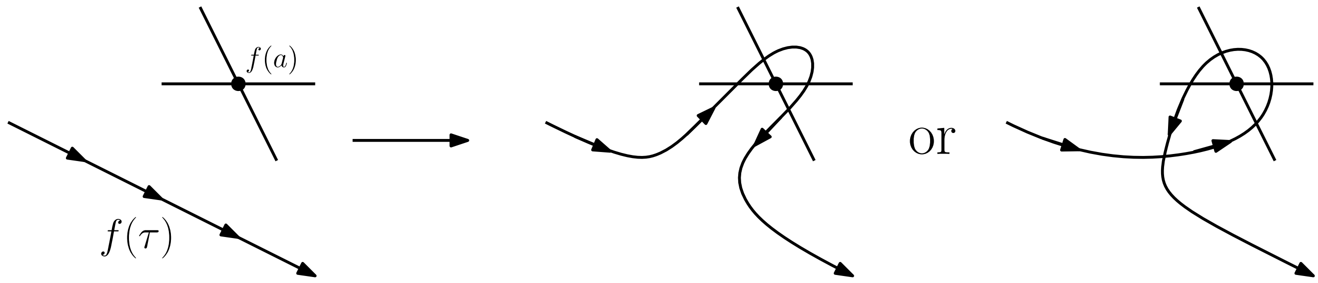

FIGURE 3.3. Finger moves (for a map f, of an edge τ, w.r.t. the vertex a) of the first and the second types respectively

Hint. Transformation of a map shown in Figure 3.3 is useful to construct examples.

For more on winding number and related notions see [Wn, Va81, To84, Ta88], [KK18, Theorem 2], [Sk18, §2].

4. Almost embeddings: definition and discussion

Theorem 4.1 (Hanani-Tutte; van Kampen). *For any map $K_5 \to \mathbb{R}^2$ there are two non-adjacent edges whose images intersect.*

This follows by Assertion 2.1.a and Theorem 5.5.a below. (The standard proof [Sk18, §1.4] does not use the winding number.) The analogue for $K_{3,3}$ holds by Assertions 2.1.a and 5.7.a below.

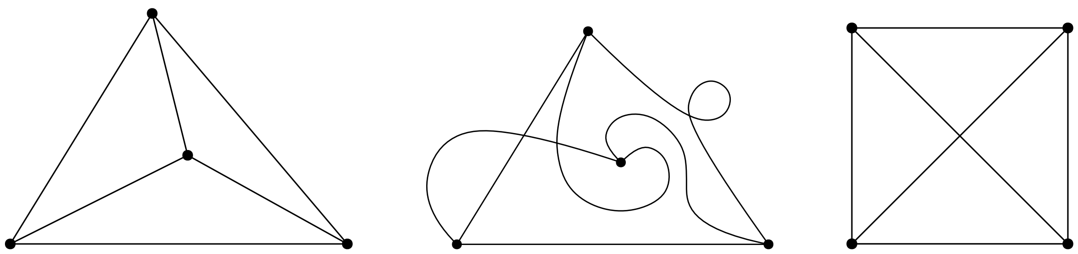

FIGURE 4.1. An embedding, an almost embedding, and a map (drawing) which is not an almost embedding

A map $f : K \to \mathbb{R}^2$ of a graph K is called an **almost embedding** if $f(\alpha) \cap f(\beta) = \varnothing$ for any two non-adjacent simplices (i.e. vertices or edges) $\alpha, \beta \subset K$. In other words, if

(i) the images of non-adjacent edges are disjoint,

(ii) the image of a vertex is not contained in the image of any edge non-adjacent to this vertex,

(iii) the images of distinct vertices are distinct.

Remark. (a) This text primarily concerns not the problem of *existence* of an almost embedding, but the *invariants* of almost embeddings. Thus we keep in the definition the properties (ii,iii) (which can be achieved by a small enough perturbation of a map, keeping the property (i)). *If a graph admits an almost embedding to the plane, then the graph is planar* (a proof is non-trivial).

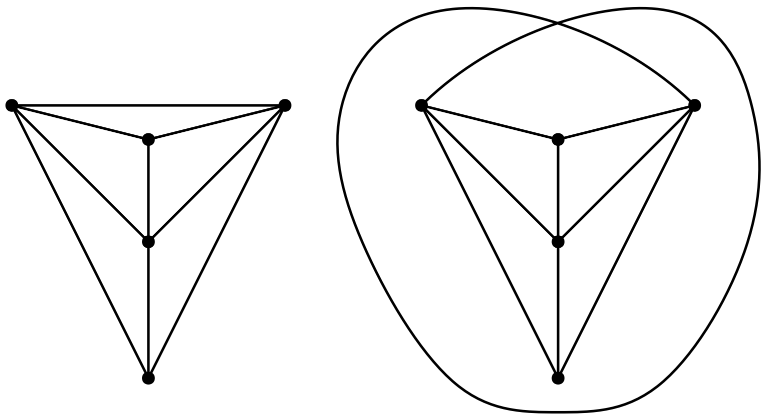

FIGURE 4.2. An embedding and an almost embedding of K_5 without an edge

(b) Almost embeddings naturally appear in topological graph theory, in combinatorial geometry, in topological combinatorics, and in studies of embeddings (of graphs in surfaces, and of hypergraphs in higher-dimensional Euclidean space). See more motivations in [ST17, §1, 'Motivation and background'], [Sk, §6.10 'Almost embeddings, $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings'].

5. MAIN RESULTS: WINDING NUMBERS OF ALMOST EMBEDDINGS

5.1. (a) For any integer n and point O in the plane there is an almost embedding $f : K_3 \to \mathbb{R}^2 - O$ such that $w(f|_{123}, O) = n$.

(b) For any integer n there is an almost embedding $f : K_4 \to \mathbb{R}^2$ such that $w_f(123, 4) = n$.

For any embedding $f : K_3 \sqcup \{4\} \to \mathbb{R}^2$ we have $w_f(123, 4) \in \{-1, 0, 1\}$ (this result is close to Jordan Curve Theorem).

Recall that

$$\sum_{j=1}^{4} w_f(C_j, j) = w_f(234, 1) + w_f(134, 2) + w_f(124, 3) + w_f(123, 4).$$

Theorem 5.2. *For any almost embedding $f : K_4 \to \mathbb{R}^2$ we have $\sum_{j=1}^{4} w_f(C_j, j) \equiv 1 \mod 2$.*

The analogue of Theorem 5.2 for embeddings instead of almost embeddings is simple (and is close to Jordan Curve Theorem). Moreover, for any embedding $f : K_4 \to \mathbb{R}^2$ three of the four numbers from Theorem 5.2 are zeroes, and the remaining one is ± 1. The analogue of Theorem 5.2 for maps instead of almost embeddings is incorrect by Assertion 3.2. Unlike Assertion 3.1, Theorem 5.2 does not come from the 'relation $123 + 134 + 142 + 243 = 0$ in the graph'.

A *general position* map $f : K_n \to \mathbb{R}^2$ is defined in [Sk18, §1.4].

Sketch of a proof of Theorem 5.2. For a general position map $f : K_4 \to \mathbb{R}^2$ let the *Radon number* $\rho(f) \in \mathbb{Z}_2$ be the the parity of the sum of

- the number of intersections points of the images of non-adjacent edges, and
- the number of vertices j whose images belong to the interior modulo 2 of $f(C_j)$.

By Assertion 2.2.a, *for every general position almost embedding $f : K_4 \to \mathbb{R}^2$ the parity of $\sum_{j=1}^{4} w_f(C_j, j)$ equals $\rho(f)$.* Theorem 5.2 is deduced using approximation from this result and the following celebrated topological Radon theorem for the plane [Sk18, Lemma 2.2.3]: *for any general position map $f : K_4 \to \mathbb{R}^2$ the Radon number $\rho(f)$ is odd.* □

Example 5.3. *(a) For any integer n there is an almost embedding $f : K_4 \to \mathbb{R}^2$ such that $w_f(C_1, 1) = n$, $w_f(C_j, j) = 0$ for every $j = 2, 3$, and $w_f(C_4, 4) = n + 1$.*

(b) For any integers n_1, n_2, n_3, n_4 such that $\sum_{j=1}^{4}(-1)^j n_j = \pm 1$, there is an almost embedding $f : K_4 \to \mathbb{R}^2$ such that $w_f(C_j, j) = n_j$ for every $j = 1, 2, 3, 4$.

Example 5.4. *(a) (E. Morozov) There is an almost embedding $f : K_4 \to \mathbb{R}^2$ such that $\sum_{j=1}^4 (-1)^j w_f(C_j, j) \neq \pm 1$.*

(b) (conjecture; see [ALM]) For any integers n_1, n_2, n_3, n_4 whose sum is odd there is an almost embedding $f : K_4 \to \mathbb{R}^2$ such that $w_f(C_j, j) = n_j$ for every $j = 1, 2, 3, 4$.

Recall that $K - e$ is the graph obtained from a graph K by deleting an edge e.

Theorem 5.5. *For any almost embedding $f : K_5 - 45 \to \mathbb{R}^2$ we have*

(a) $w_f(123, 4) - w_f(123, 5) \equiv 1 \mod 2$;

(b) $w_f(123, 4) - w_f(123, 5) = \pm 1$.*

Cf. Theorem 5.2 and Conjecture 5.4.b. The analogue of (b) for embeddings instead of almost embeddings is simple (and is close to Jordan Curve Theorem). Part (a) is not hard and is well-known, while part (b) is a recent non-trivial result of [Ga23].

5.6. (a) Is the analogue of Theorem 5.5.a correct for maps instead of almost embeddings?

(b) For any integer n there is an almost embedding $f : K_5 - 45 \to \mathbb{R}^2$ such that $w_f(123, 5) = n$. (See Figure 4.2, right, for $n = 2$.)

Sketch of a proof of Theorem 5.5.a. For a general position map $f : K_5 \to \mathbb{R}^2$ color in red the intersections points of the images of non-adjacent edges. Let *the van Kampen number* $v(f) \in \mathbb{Z}_2$ be the parity of the number of red points. By Assertion 2.2.a, *for any general position map $f : K_5 \to \mathbb{R}^2$ whose restriction to $K_5 - 45$ is an almost embedding the parity of $w_f(123, 5) - w_f(123, 4)$ equals $v(f)$.* Theorem 5.5.a is deduced from this and the following celebrated van Kampen-Flores theorem for the plane [Sk18, Lemma 1.4.3]: *for any general position map $f : K_5 \to \mathbb{R}^2$ the van Kampen number $v(f)$ is odd.* [4] $\square$

5.7. Take an edge ab of $K_{3,3}$. Denote by $C = C_{ab}$ somehow oriented cycle $K_{3,3} - a - b$ of length 4. For any almost embedding $f : K_{3,3} - ab \to \mathbb{R}^2$ we have

(a) $w_f(C, a) - w_f(C, b) \equiv 1 \mod 2$; (b) (conjecture) $w_f(C, a) - w_f(C, b) = \pm 1$.

Part (a) is proved analogously to Theorem 5.5.a. The analogue of (b) for embeddings instead of almost embeddings is simple (and is close to Jordan Curve Theorem). Beware that a direct proof of (b) might contains technical details (like for Theorem 5.5.b); perhaps there is a simple a reduction to Theorem 5.5.b, see Figure 5.1.

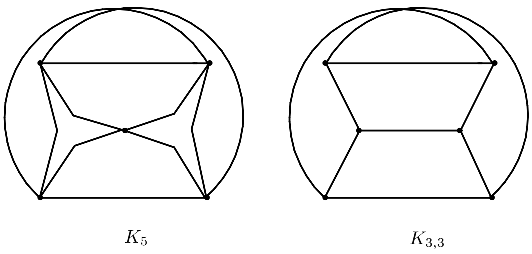

FIGURE 5.1. 'Almost embedding' $K_5 \to K_{3,3}$

5.8 (open problem; riddle)**.** Let K be the graph of

(a) a cube; (b) an octahedron.

For an almost embedding $f : K \to \mathbb{R}^2$ consider the collection $w_f(C, v)$ of integers, where $v \in K$ is a vertex, and $C \subset K - v$ is an oriented cycle. Describe collections realizable by almost embeddings $f : K \to \mathbb{R}^2$.

[4] Theorem 5.5.b is an *integer version for almost embeddings* of this theorem. Observe that this theorem has no *integer version for maps* (this is known and is explained in [Ga23, Remark 4]).

Remark. (a) The integer $w_f(C, v)$ and invariants studied in §6 are *(almost) isotopy invariants* of an (almost) embedding $f : K \to \mathbb{R}^2$. They are parts of the *Haefliger-Wu invariant* of f [Sk06, §5].

(b) An algebraic version of almost embedings ($\mathbb{Z}_2$-embeddings) appeared in 1930s and is actively studied in graph theory since 2000s. See e.g. surveys [SS13], [Sk, §6.10 'Almost embeddings, $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddings'], and the paper [FK19] relating $\mathbb{Z}_2$-embeddings to low rank matrix completion problem. The analogues of assertions 5.2, 5.5.a, 5.7.a are correct for $\mathbb{Z}_2$-embeddings. We conjecture that the analogues of assertions 5.5.b, 5.7.b are correct for $\mathbb{Z}$-embeddings, but are incorrect for $\mathbb{Z}_2$-embeddings.

(c) A *hypergraph* is a higher-dimensional analogue of graph: together with edges joining pairs of points one considers triangles spanned by triples of points, etc. A classical problem in topology, combinatorics and computer science is to find criteria (and algorithms) for realizability of hypergraphs in Euclidean space of given dimension d.

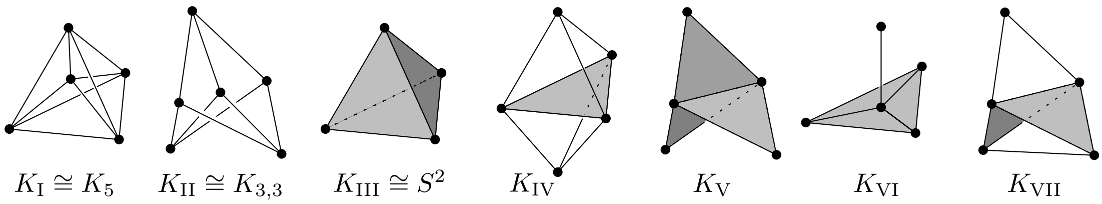

FIGURE 5.2. Two-dimensional hypergraphs non-embeddable in the plane

Such a criterion was obtained in 1930s-1960s by classical figures in topology. The criterion is stated in terms of certain configuration space, yielded many specific corollaries, and works for $2d \geq 3k + 3$, where k is the dimension of the hypergraph. A polynomial algorithm based on this criterion was obtained in 2013. The non-existence of a polynomial algorithm for $2d < 3k + 2$ was announced in 2019 by Marek Filakovský, Ulrich Wagner and Stephan Zhechev. A mistake was found in 2020 by Arkadiy Skopenkov (and recognized by the authors). The mistake was that in a higher-dimensional analogue of Theorem 5.5.b (and of Example 7.1.a) certain linking invariant can assume values distinct from ± 1. In 2020 Roman Karasev and Arkadiy Skopenkov showed that the linking invariant can assume any odd value. Their conjecture that the same holds for graphs in the plane was refuted by Timur Garaev, see Theorem 5.5.b. For references see surveys [Sk06, §5], [Sk18, §3], and recent research papers [KS20, Ga23].

6. Triodic and cyclic Wu numbers

In the plane let l_1, l_2, l_3 be polygonal lines joining a point O to points A_1, A_2, A_3, respectively. Assume that $A_i \notin l_j$ for every $i \neq j$. (In other words, recall that $K_{3,1}$ is the graph with vertices $\{1, 2, 3, 1'\}$, where $\deg 1' = 3$ and $\deg m = 1$ for each $m \in [3]$; take an almost embedding $f : K_{3,1} \to \mathbb{R}^2$ and denote $l_m := f(1'm)$ for each $m \in [3]$.)

The *triodic Wu number* $\mathrm{wu}(l_1, l_2, l_3)$ is defined to be the number of revolutions in the following rotation of vector:

- from $\overrightarrow{A_1A_2}$ to $\overrightarrow{A_1A_3}$ as the second point of the vector moves along $\overline{l_2}l_3$, then
- from $\overrightarrow{A_1A_3}$ to $\overrightarrow{A_2A_3}$ as the first point of the vector moves along $\overline{l_1}l_2$, then
- from $\overrightarrow{A_2A_3}$ to $\overrightarrow{A_2A_1}$ as the second point of the vector moves along $\overline{l_3}l_1$, then
- from $\overrightarrow{A_2A_1}$ to $\overrightarrow{A_3A_1}$ along $\overline{l_2}l_3$, then
- from $\overrightarrow{A_3A_1}$ to $\overrightarrow{A_3A_2}$ along $\overline{l_1}l_2$, then
- from $\overrightarrow{A_3A_2}$ to $\overrightarrow{A_1A_2}$ along $\overline{l_3}l_1$.

This equals twice the (non-integer) number of revolutions in first three rotations above. Rigorously,

$$\mathrm{wu}(l_1, l_2, l_3) := w'(\overline{l_2}l_3, A_1) + w'(\overline{l_1}l_2, A_3) + w'(\overline{l_3}l_1, A_2) + w'(\overline{l_2}l_3, A_1) + w'(\overline{l_1}l_2, A_3) + w'(\overline{l_3}l_1, A_2) =$$

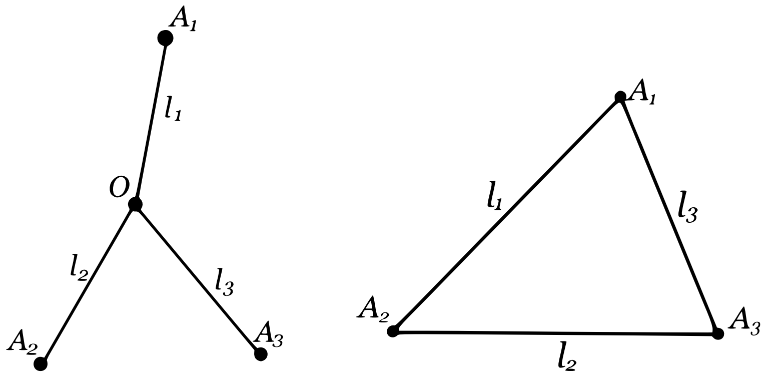

FIGURE 6.1. A triod and a triangle

$$= 2\left(w'(\overline{l_2}l_3, A_1) + w'(\overline{l_1}l_2, A_3) + w'(\overline{l_3}l_1, A_2)\right). \qquad (*)$$

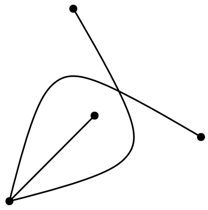

FIGURE 6.2. Three polygonal lines whose triodic Wu number equals ± 3

6.1. (a) For the three segments joining the vertices A_1, A_2, A_3 of a regular triangle to its center O, the triodic Wu number equals ± 1.

(b) For the three polygonal lines shown in Figure 6.2 the triodic Wu number equals ± 3.

6.2. (a) For any polygonal lines l_1, l_2, l_3 as above (i.e. for any an almost embedding $f : K_{3,1} \to \mathbb{R}^2$) the triodic Wu number is odd.

(b) For any integer n there are polygonal lines l_1, l_2, l_3 as above (i.e. there is an almost embedding $f : K_{3,1} \to \mathbb{R}^2$) whose triodic Wu number equals $2n + 1$.

(c) For any embedding $f : K_{3,1} \to \mathbb{R}^2$ the triodic Wu number is ± 1.

(d) Permutation of the polygonal lines l_1, l_2, l_3 as above multiplies the triodic Wu number by the sign of the permutation.

In the plane let A_1, A_2, A_3 be points and l_1, l_2, l_3 polygonal lines joining A_1 to A_2, A_2 to A_3, A_3 to A_1, respectively (and thus forming a closed polygonal line). Assume that A_i is not contained in l_{i+1} for each $i = 1, 2, 3$ (the numeration is modulo 3; in other words, the polygonal lines form an almost embedding $K_3 \to \mathbb{R}^2$). The *cyclic Wu number* $\mathrm{wu}(l_1, l_2, l_3)$ is defined to be twice the number of revolutions in the following rotation of vector:

- from $\overrightarrow{A_1A_2}$ to $\overrightarrow{A_1A_3}$, as the second point of the vector moves along l_2, then
- from $\overrightarrow{A_1A_3}$ to $\overrightarrow{A_2A_3}$, as the first point of the vector moves along l_1, then
- from $\overrightarrow{A_2A_3}$ to $\overrightarrow{A_2A_1}$, as the second point of the vector moves along l_3.

In other words, $\mathrm{wu}(l_1, l_2, l_3)$ is defined by the following formula analogous to (*):

$$\mathrm{wu}(l_1, l_2, l_3) := 2\left(w'(l_2, A_1) + w'(l_1, A_3) + w'(l_3, A_2)\right). \qquad (**)$$

6.3. (a') If the three polygonal lines l_1, l_2, l_3 as above are sides of a triangle, then the cyclic Wu number is ± 1.

(a) For any polygonal lines l_1, l_2, l_3 as above the cyclic Wu number is odd.

(b) For any integer n there are polygonal lines l_1, l_2, l_3 as above whose cyclic Wu number equals $2n + 1$.

If three polygonal lines as above form a simple closed polygonal line, then the cyclic Wu number is ±1 (this is close to Jordan Curve Theorem).

The cyclic Wu number is similar to, but distinct from, the *degree* of a closed curve.

6.4. (a) For any almost embedding $f : K_4 \to \mathbb{R}^2$ we have

$$\mathrm{wu}(f|_{12}, f|_{23}, f|_{31}) + \mathrm{wu}(f|_{41}, f|_{42}, f|_{43}) = 2w_f(123, 4).$$

(b) (conjecture; see [Za]) For any two odd integers n, m there is an almost embedding $f : K_4 \to \mathbb{R}^2$ such that $\mathrm{wu}(f|_{12}, f|_{23}, f|_{31}) = m$ and $\mathrm{wu}(f|_{41}, f|_{42}, f|_{43}) = n$.

6.5. (a) For any almost embedding $f : K_5 - 45 \to \mathbb{R}^2$ we have

$$\mathrm{wu}(f|_{41}, f|_{42}, f|_{43}) - \mathrm{wu}(f|_{51}, f|_{52}, f|_{53}) = 2(w_f(123, 4) - w_f(123, 5)).$$

(b) (conjecture) For any almost embedding $f : K_{3,2} \to \mathbb{R}^2$ we have

$$\mathrm{wu}(f|_{1'1}, f|_{1'2}, f|_{1'3}) - \mathrm{wu}(f|_{2'1}, f|_{2'2}, f|_{2'3}) = \pm 2.$$

By (a), a simple proof of (b) would give a simple proof of Theorem 5.5.b.

7. 3-dimensional analogues

The Linear Conway–Gordon–Sachs Theorem. If no 4 of 6 points in 3-space lie in one plane, then there are two linked triangles with vertices at these 6 points (i.e. the first triangle intersects the outline of the second triangle exactly at one point). For a proof see survey [Sk14].

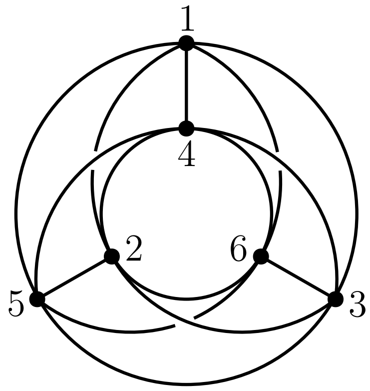

Figure 7.1. A projection on the plane of an embedding $K_6 \to \mathbb{R}^3$

Equivalent rigorous definitions of a *linking number* of disjoint closed polygonal lines in 3-space can be found in [Sk20u, §§4,8], [Sk24, §§1.2,1.3], [Sk, §§4.2,4.3].

Example 7.1 (cf. Figure 7.1). * *(a) For any integer n there are six points in 3-space, and non-self-intersecting polygonal lines joining each pair of them, and such that*

(a1) the interior of one polygonal line is disjoint with any other polygonal line,

(a2) the linking number of one (unordered) pair of disjoint cycles of length 3 formed by the polygonal lines is $2n + 1$, and

(a3) the linking number of any other pair of disjoint cycles of length 3 formed by the polygonal lines is zero.

(b) (conjecture) Take any 10 integers $n_{123,456}, n_{124,356}, \ldots$ corresponding to the 10 non-ordered splittings of $[6]$ into two 3-element subsets. If the sum of the integers is odd, then there are 6 points $1, 2, 3, 4, 5, 6$ in 3-space, and non-self-intersecting polygonal lines joining each pair them, for which (a1) holds, and the linking number of every pair $\{ijk, pqr\}$ of disjoint cycles of length 3 formed by the polygonal lines equals $n_{ijk,pqr}$.

Part (a) is proved in [KS20, Proposition 1.2] but could have been known before. Part (b) could perhaps be proved using [KS20, Lemma 2.1]. For higher-dimensional analogues see survey [Sk14] and [KS20, §1]. In those results the linking number can assume any odd value, like Conjecture 5.4.b, but unlike Conjecture 5.7.b and Theorem 5.5.b.

8. Answers, hints and solutions

1.1. For $j = 1, \dots, m$ let $t_j := \angle A_j O A_{j+1}$, where $A_{m+1} = A_1$. Then

$$\overrightarrow{OA_1} \upuparrows e^{it_m}\overrightarrow{OA_m} \upuparrows e^{i(t_m+t_{m-1})}\overrightarrow{OA_{m-1}} \upuparrows \dots \upuparrows e^{i(t_m+t_{m-1}+\dots+t_1)}\overrightarrow{OA_1}.$$

Hence $(t_m + t_{m-1} + \dots + t_1)/2\pi$ is an integer.

1.2. (a) Let Ω be a convex polygon.

If O is in the exterior of Ω, draw two supporting lines from O to Ω. Take two points A, B from intersections of the lines and $\partial\Omega$. Then $w(\partial\Omega, O) = \frac{1}{2\pi}(\angle AOB + \angle BOA) = 0$.

If O is in the interior of Ω, draw a regular triangle ABC centered at O. Take three intersection points A', B', C' of the rays OA, OB, OC with $\partial\Omega$. They split $\partial\Omega$ in three polygonal lines. We have

$$w(\partial\Omega, O) = \frac{1}{2\pi}(\angle A'OB' + \angle B'OC' + \angle C'OA') = \frac{3}{2\pi}\angle A'OB' = \frac{3}{2\pi}\angle AOB = \pm 1.$$

(b) By (a), we have $w(ABCABC, O) = 2 \cdot w(ABC, O) = \pm 2$.

(c) If $n = 0$, example is a single-point closed polygonal line. If $n \neq 0$, let ABC be a regular triangle oriented clockwise if $n < 0$, and counterclockwise in the opposite case. Let O be its center. Consider the closed polygonal line $L = \underbrace{ABC\dots ABC}_{|n| \text{ times}}$. By a refinement of (a), we have $w(L, O) = |n| \cdot w(ABC, O) = n$.

(d) A trivial example is a single-point closed polygonal line. Another example: a closed polygonal line $ABCB$ for any three points A, B and C in the plane.

1.3. By the symmetry, $w'(A_1 \dots A_{k+1}, O) = w'(A_{k+1} \dots A_{2k}A_1, O)$. Then

$$w(A_1 \dots A_{2k}, O) = w'(A_1 \dots A_{k+1}, O) + w'(A_{k+1} \dots A_{2k}A_1, O) =$$
$$= 2w'(A_1 \dots A_{k+1}, O) = 2\left(\frac{\angle A_1OA_{k+1}}{2\pi} + n\right) = 1 + 2n$$

for some integer n. Here the last but one equality follows from Assertion 1.4.a.

1.4. (a) Using the formula from the proof of Assertion 1.1, we have

$$e^{i\angle A_1OA_m}\overrightarrow{OA_1} \upuparrows \overrightarrow{OA_m} \upuparrows e^{2\pi i w'(A_1\dots A_m,O)}\overrightarrow{OA_1}.$$

Hence $\angle A_1OA_m - 2\pi w'(A_1 \dots A_m, O) = 2\pi k$ for some integer k.

1.5. *Hint.* (a) Use Assertion 1.4.b and the following equality: $w'(l, O) = -w'(\bar{l}, O)$ for every point O and every polygonal line l not passing through O.

1.6. *Hint.* We can assume that the vertices of the triangle are numbered counter-clockwise. Prove that $w'(l_0, O) = w'(l_1, O) = w'(l_2, O) = \frac{2\pi}{3}$. For that denote by $B_1 \dots B_m$ the sequence of vertices of l_0. For $j \in [m-1]$ let $t_j := \angle B_jOB_{j+1} \in (-\pi, \pi)$. Prove that for every $j \in [m-1]$ we have $T_j := t_1 + \dots + t_j \in (-\frac{2\pi}{3}, \frac{4\pi}{3})$, and $w'(l_0, O) = T_{m-1} = \frac{2\pi}{3} + 2\pi k$ for some integer k. Then deduce that $k = 0$.

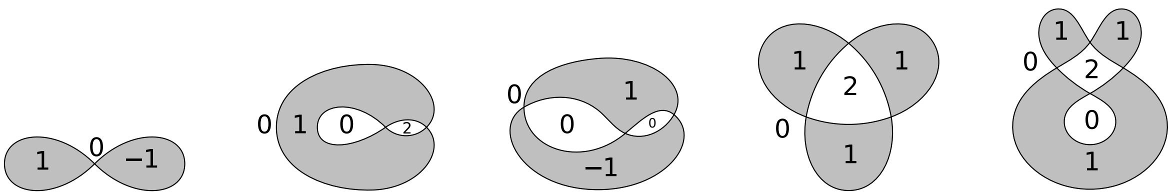

Figure 8.1. Coloring of the complement of the closed polygonal lines from Figure 2.1 by the winding number

2.1. (b,c) Figure 8.1.

3.1. *Hint.* Use the following equalities: $w'(f|_{ij}, O) = -w'(f|_{ji}, O)$ for every edge ij, and $w(f|_{ijk}, O) = w'(f|_{ij}, O) + w'(f|_{jk}, O) + w'(f|_{ki}, O)$ for every cycle ijk.

3.2. *Hints.* (a) A square with the diagonals forms a map $f : K_4 \to \mathbb{R}^2$; assume the vertices $f(i)$, $i \in [4]$ are numbered counterclockwise. Then $w_f(C_j, j) = 0$ for every $j \in [4]$. Make $|n|$ finger moves (Figure 3.3) of the edge 24 w.r.t. the vertex 1 of the first/second type if n is positive/negative respectively. The obtained map f_1 is as required.

(b) Consider the map f_1 from the proof of (a) for $n = n_1$. Make $|n_2|$ finger moves of the edge 13 w.r.t. the vertex 2 of the first/second type if n_2 is positive/negative respectively. Denote the obtained map by f_2. Make $|n_3|$ finger moves of the edge 24 w.r.t. the vertex 3 of the first/second type if n_3 is positive/negative respectively. Denote the obtained map by f_3. Make $|n_4|$ finger moves of the edge 13 w.r.t. the vertex 4 of the first/second type if n_4 is positive/negative respectively. The obtained map f_4 is as required.

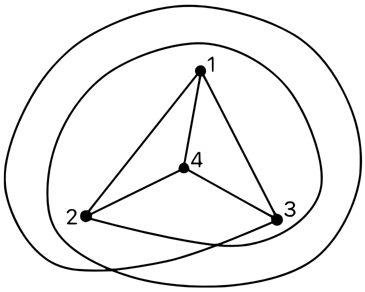

FIGURE 8.2. An almost embedding $f : K_4 \to \mathbb{R}^2$ such that $w_f(123, 4) = 3$

5.1. (b) See Figure 8.2 for $n = 3$. To obtain a map for (a) remove the images of edges issuing out of $f(4)$, and replace $f(4)$ by O.

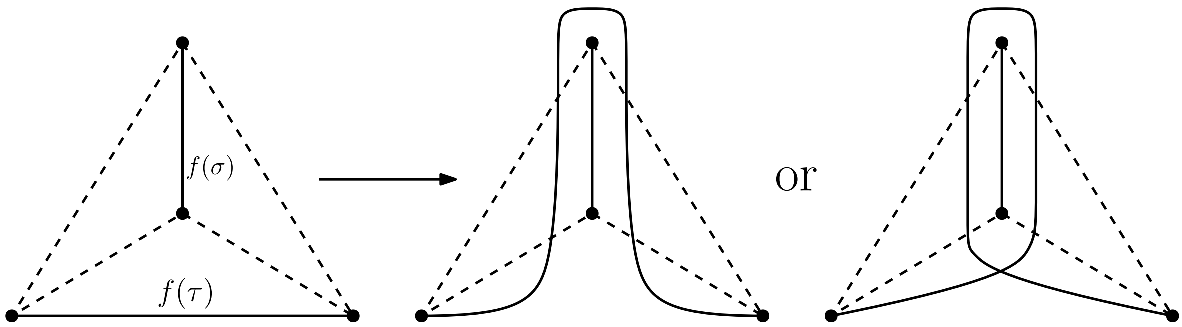

FIGURE 8.3. Finger moves (for a map f, of an edge τ, w.r.t. the segment $f(\sigma)$) of the first and the second types respectively

5.3. *Hints.* (a) A regular triangle with its center and edges connecting the center to the vertices form a map $f : K_4 \to \mathbb{R}^2$; assume $f(1), f(2), f(3)$ are the vertices numbered counterclockwise. Then $w_f(C_j, j) = 0$ for every $j \in [3]$, and $w_f(C_4, 4) = 1$. Make $|n|$ finger moves of the edge 23 w.r.t. the segment $f(14)$ (Figure 8.3, cf. Figure 3.3) of the first/second type if n is negative/positive respectively. The obtained map f_1 is as required.

(b) In the next paragraph we construct an almost embedding for $\sum_{j=1}^{4}(-1)^j n_j = 1$. To construct an almost embedding f for $\sum_{j=1}^{4}(-1)^j n_j = -1$ take an almost embedding g for $m_1 = n_2$, $m_2 = n_1$, $m_3 = -n_3$, $m_4 = -n_4$, where $\sum_{j=1}^{4}(-1)^j m_j = 1$. Then $f = g \circ \sigma$, where $\sigma : K_4 \to K_4$ is a permutation that interchanges the vertices 1 and 2.

Consider the map f_1 from the proof of (a) for $n = n_1$. Make $|n_2|$ finger moves of the edge 13 w.r.t. the segment $f_1(24)$ of the first/second type if n_2 is positive/negative respectively. Denote the obtained map by f_2. Make $|n_3|$ finger moves of the edge 12 w.r.t. the segment $f_2(34)$ of the first/second type if n_3 is negative/positive respectively. Denote the obtained map by f_3. The obtained map f_3 is as required.

5.4. *Hint.* (a) See Figure 8.4.

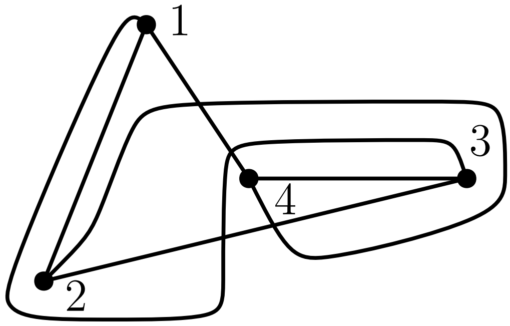

FIGURE 8.4. An almost embedding $f : K_4 \to \mathbb{R}^2$ such that $\sum_{j=1}^{4}(-1)^j w_f(C_j, j) = 3$

5.6. (a) No. A regular pentagon $f(1)\dots f(5)$ with all the diagonals, but without the edge $f(4)f(5)$ forms a map $f : K_5 - 45 \to \mathbb{R}^2$ such that $w_f(123, 4) - w_f(123, 5) = 0$.

(b) See Figure 8.5 for $n = 3$.

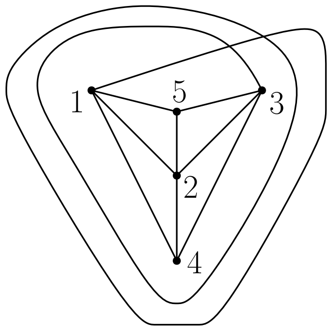

FIGURE 8.5. An almost embedding of K_5 without an edge 45 such that $w_f(123, 5) = 3$

6.2. (a) We have

$$\mathrm{wu}(l_1, l_2, l_3) \overset{(1)}{=} 2\left(\frac{\angle A_2A_1A_3}{2\pi} + k_1 + \frac{\angle A_3A_2A_1}{2\pi} + k_2 + \frac{\angle A_1A_3A_2}{2\pi} + k_3\right) =$$

$$= 2(k_1 + k_2 + k_3) + 2\frac{\angle A_2A_1A_3 + \angle A_3A_2A_1 + \angle A_1A_3A_2}{2\pi} = 2(k_1 + k_2 + k_3) + 1$$

for some integers k_1, k_2, k_3. Here equality (1) follows by (*) and Assertion 1.4.a.

(b) Consider segments l_1, l_2, l_3 having a common point O in the interior of triangle $A_1A_2A_3$ such as Figure 6.1. Make $|n|$ finger moves (Figure 3.3) of the edge l_1 w.r.t. the vertex A_3 of the first/second type for negative/positive n respectively. Then

$$\mathrm{wu}(l_1, l_2, l_3) = 2\left(\frac{\angle A_2A_1A_3}{2\pi} + \frac{\angle A_3A_2A_1}{2\pi} + \frac{\angle A_1A_3A_2}{2\pi} + n\right) = 1 + 2n.$$

For example, in Figure 8.6, left,

$$w'(\overline{l_2}l_3, A_1) = \frac{\angle A_2A_1A_3}{2\pi}, \quad w'(\overline{l_3}l_1, A_2) = \frac{\angle A_3A_2A_1}{2\pi}, \quad w'(\overline{l_1}l_2, A_3) = \frac{\angle A_1A_3A_2}{2\pi} + 2,$$

so $\mathrm{wu}(l_1, l_2, l_3) = 2 \cdot 2 + 1 = 5$.

(c) *Hint.* This is proved by induction on the number of segments in $l_1 \cup l_2 \cup l_3$.

6.3. (a) We have

$$\mathrm{wu}(l_1, l_2, l_3) \overset{(1)}{=} 2\left(\frac{\angle A_2A_1A_3}{2\pi} + k_1 + \frac{\angle A_3A_2A_1}{2\pi} + k_2 + \frac{\angle A_1A_3A_2}{2\pi} + k_3\right) =$$

$$= 2(k_1 + k_2 + k_3) + 2\frac{\angle A_2A_1A_3 + \angle A_3A_2A_1 + \angle A_1A_3A_2}{2\pi} = 2(k_1 + k_2 + k_3) + 1$$

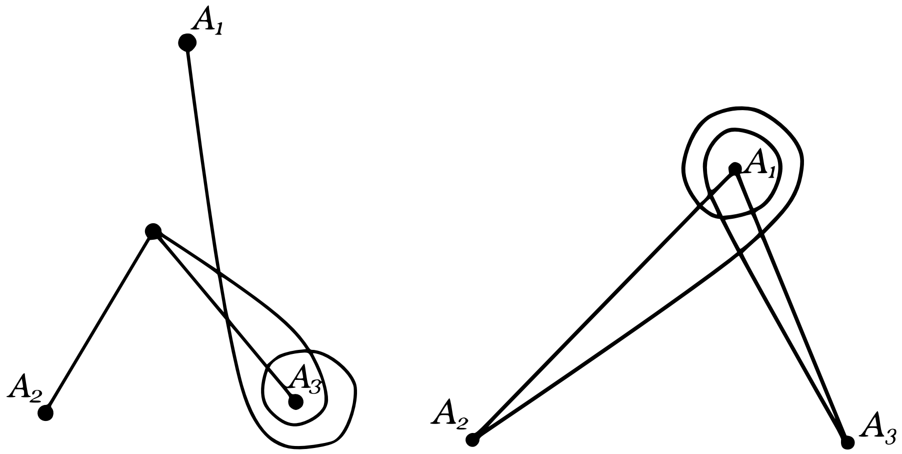

FIGURE 8.6. Three polygonal lines whose triodic/cyclic (left/right) Wu number equals 5

for some integers k_1, k_2, k_3. Here equality (1) follows by (**) and Assertion 1.4.a.

(b) Consider triangle $A_1A_2A_3$ with sides l_2, l_3, l_1, see Figure 6.1. Make $|n|$ finger moves (Figure 3.3) of the edge l_2 w.r.t. the vertex A_1 of the first/second type for negative/positive n respectively. Then

$$\mathrm{wu}(l_1,l_2,l_3) = 2\left(\frac{\angle A_2A_1A_3}{2\pi} + n + \frac{\angle A_3A_2A_1}{2\pi} + \frac{\angle A_1A_3A_2}{2\pi}\right) = 2n+1.$$

For example, in Figure 8.6, right,

$$w'(l_2,A_1) = \frac{\angle A_2A_1A_3}{2\pi} + 2, \quad w'(l_3,A_2) = \frac{\angle A_3A_2A_1}{2\pi}, \quad w'(l_1,A_3) = \frac{\angle A_1A_3A_2}{2\pi},$$

so $\mathrm{wu}(l_1,l_2,l_3) = 2\cdot 2+1 = 5$.

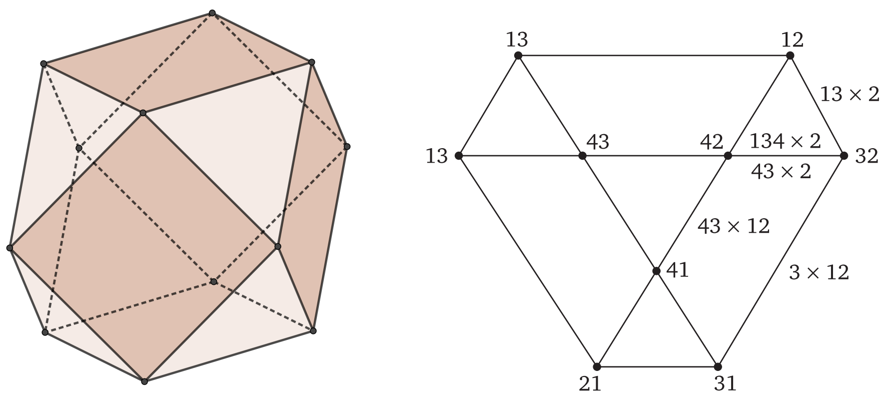

FIGURE 8.7. Left: a magic cuboctahedron. Right: the magic: the figure does not show the invisible part whose projection is obtained from the pictured projection by rotation through $\pi/3$; the lower 13 should be 23; 4×123 is the central triangle, 123×4 is the invisible central triangle, 24×13 is the bottom left trapezoid, 13×24 is the upper right invisible trapezoid, $\mathrm{off}(123) = 12, 13, 23, 21, 31, 32$ is the outer cycle of length 6, $\mathrm{triod}(123,4) = 12, 14, 13, 43, 23, 24, \dots$, where dots denote the part symmetric to the written part (obtained replacing xy by yx) (each of the cycles $\mathrm{off}(123)$ and $\mathrm{triod}(123,4)$ splits the cuboctahedron into two equal parts)

6.4. *Hint:* see Figure 8.7 and [ADN+, §2, §5]; $\mathrm{off}(123) + \mathrm{triod}(123,4) = 123\times 4 + 4\times 123 + \partial(13\times 24) + \partial(24\times 13) + \partial(12\times 34) + \partial(34\times 12) + \partial(14\times 23) + \partial(23\times 14)$.

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