Deviations of polynomials and critical values

Project team:

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In this project some properties of the polynomials are discussed. You can use without restrictions theorems from «introductory» course of Calculus, in particular, the following theorems.

Bolzano-Cauchy's theorem (Intermediate value theorem). If continuous function f takes on the endpoints of segment $[a, b]$ the values of different signs, then it has a root on this segment.

Weierstrass theorem (Extreme value theorem). Any continuous function f defined on segment $[a, b]$ is bounded on it and attains the largest and the smallest values.

1 Several preliminary problems

In this section a few problems are given for acquaintance with project. Not all of these problems are simple and it is not necessary to solve them «first of all».

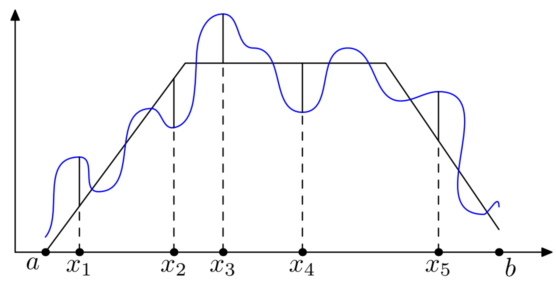
1.1. For an arbitrary polynomial $F(x)$ set

$$F^{[n]}(x) = \underbrace{F(F(\ldots F}_{n\text{ times}}(x))).$$

Prove that there is cubic polynomial $P(x)$ such that for each positive integers N the following equations have 3^N distinct real roots in $[-1, 1]$. a) $P^{[N]}(x) = 0$; b) $P^{[N]}(x) = x$.

1.2. Let $|ax^2 + bx + c| \leqslant 1$ for all $|x| \leqslant 1$. Prove that $|2ax + b| \leqslant 4$, for all $|x| \leqslant 1$.

1.3. Let n be a natural number and a continuous function f is defined on the segment $[a, b]$ (we can suppose for simplicity that it is piecewise linear, as in the figure). For each polynomial F of degree n denote



$$M(F) = \max_{x\in[a,b]} |F(x) - f(x)|.$$

Suppose that there exists polynomial F_n of degree n, for which $M(F_n)$ attains the minimum possible value. Prove that on segment $[a, b]$ there exist points $a \leqslant x_1 < x_2 < \ldots < x_{n+1} \leqslant b$ such that for all k

$$F_n(x_k) - f(x_k) = \pm M(F_n),$$

and for any pair of consecutive points x_k, x_{k+1} the differences $F_n(x) - f(x)$ have the opposite signs.

1.4. Prove that for any n given points $A_1, A_2, \ldots, A_n$ on the plane, the product of the distances

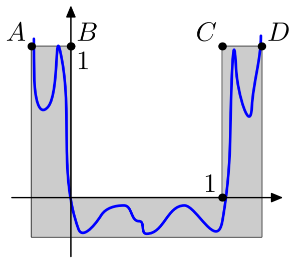
$$MA_1 \cdot MA_2 \cdot \ldots \cdot MA_n$$

from them to point M running the given segment $[a, b]$ of length $b - a = 2h$, cannot remain all the time less than $2\left(\frac{h}{2}\right)^n$.

1.5. Problem «about the glass». We call «glass» the subset on Cartesian coordinate system drawn in the figure. The width of vertical walls and of the bottom is equal to δ. Do arbitrary large n exist such that the graph of a polynomial of degree n lies inside the «glassy» part of the glass), (this part is shaded in the figure) if

a) $\delta = \frac{1}{n}$ b) $\delta = \frac{1}{n^3}$?

The graph must come into the shaded region intersecting segment AB, and leave this region through CD.



2 Polynomials that have small deviation from 0

By *deviation* (from 0) of a polynomial F on segment $[a, b] \subset \mathbb{R}$ we call the value

$$M(F) = \max_{x \in [a,b]} |F(x)|.$$

We say that polynomial F has *small deviation*, if $M(F) \leqslant 1$. Remind that polynomial is called *monic*, if its leading coefficient is equal to 1. In this project we consider polynomials only with real coefficients (but in the solutions you can use complex numbers).

Now we will prove that among all monic polynomials of degree n there exists polynomial F_n, the deviation of which $M(F_n)$ takes the minimum possible value on segment $[-1, 1]$.

Suppose that for some number c we found monic polynomial F_n, for which $M(F_n) = c$ and (as in problem 1.3) *there exist such points* $-1 \leqslant x_1 < x_2 < \ldots < x_{n+1} \leqslant 1$ *that for all* k

$$F_n(x_k) = \pm M(F_n),$$

and for any pair of consecutive points x_k, x_{k+1} *values* $F_n(x_k)$ *and* $F_n(x_{k+1})$ *have opposite signs.* Verify that then c is the minimum possible deviation for all monic polynomials of degree n.

Indeed, suppose that polynomial Q with the lesser deviation has been found. Then for $x \in [-1, 1]$ the graph of polynomial $F_n(x)$ lies in the horizontal stripe of width $2c$ and comes on its boundary in points $(x_k, F_n(x_k))$, and the graph of polynomial Q lies strictly inside the stripe, see fig. 1. Draw through points x_k vertical (dashed) lines, they will cut n rectangles from the stripe. Inside each rectangle the graphs of P_n and Q have at least one intersection point. But it is impossible because polynomial $P_n - Q$ has degree not less than $n-1$ and cannot have n roots.

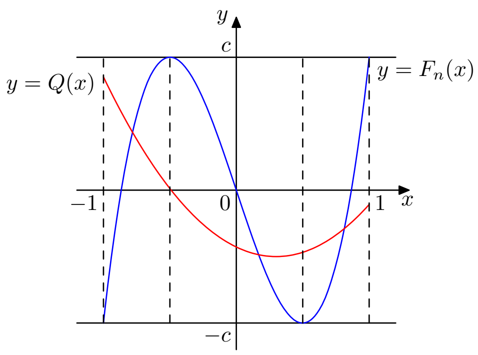


Рис. 1.

n	$T_n(x)$
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	$32x^6 - 48x^4 + 18x^2 - 1$

Рис. 2. Chebyshev's polynomials

Furthermore, in the described situation F_n is a unique monic polynomial, the deviation of which is equal to c.

Therefore, it remains «guess right» the polynomial satisfying the above property set out in italics. This construction is well known. Note that for natural n function $\cos nx$ may be expressed through $\cos x$ by trigonometric transformations and the obtaining formula is polynomial: $\cos 2x = 2(\cos x)^2 - 1$, $\cos 3x = 4(\cos x)^3 - 3\cos x$ etc. Hence, function

$$T_n = \cos(n \arccos x)$$

is polynomial (of degree n). It is called *Chebyshev polynomial* of the first kind. It is immediately clear by the definition that $|T_n(x)| \leqslant 1$ for $x \in [-1,1]$ and

$$T_n(x_k) = (-1)^k, \quad \text{for } x_k = \cos\frac{k\pi}{n}, \qquad k = 0, 1, \ldots, n.$$

Therefore, $M(T_n) = 1$, and $F_n = \frac{1}{2^{n-1}} T_n$ must be taken as polynomials F_n from the previous reasoning, and $c = M(F_n) = \frac{1}{2^{n-1}}$.

The first few Chebyshev's polynomials are given in the table (fig. 2). We give several useful formulas.

$$T_n(x) = 2^{n-1}\Big(x - \cos\frac{\pi}{2n}\Big)\Big(x - \cos\frac{3\pi}{2n}\Big)\Big(x - \cos\frac{5\pi}{2n}\Big)\ldots\Big(x - \cos\frac{(2n-1)\pi}{2n}\Big) \quad \text{for } x \in \mathbb{R}, \tag{1}$$

$$T_n(x) = \frac{1}{2}\big(x + \sqrt{x^2-1}\big)^n + \frac{1}{2}\big(x - \sqrt{x^2-1}\big)^n \qquad \text{for } x \in \mathbb{R} \setminus (-1,1). \tag{2}$$

$$2T_n(x) = (2x)^n - \frac{n}{n-1}\cdot C_{n-1}^1 (2x)^{n-2} + \frac{n}{n-2}\cdot C_{n-2}^2 (2x)^{n-4} - \frac{n}{n-3}\cdot C_{n-3}^3 (2x)^{n-6} + \ldots \tag{3}$$

2.1. Prove that polynomial F_n from the given above reasoning is uniquely defined.

2.2. Let $P(x) = a_0 + a_1 x + \cdots + a_d x^d$ be a polynomial of degree d, then

$$\max_{x\in[a,b]} |P(x)| \geqslant \frac{|a_d|}{2^{2d-1}}(b-a)^d.$$

Moreover, the equality case occurs whenever, $P(x) = \frac{a_d}{2^{2d-1}}(b-a)^d \cdot T_d\Big(\frac{2x-a-b}{b-a}\Big)$. And also prove that

$$P(x) = \frac{a_d}{2^{d-1}} \cdot \sum_{k=0}^{\lfloor d/2 \rfloor} C_d^{2k} (x-a)^k (x-b)^k \Big(x - \frac{a+b}{2}\Big)^{d-2k}$$

.

2.3. Prove the following extremal property of Chebyshev's polynomials. Let $F_n(x)$ be the polynomail of degree not more than n, and

$$\max_{x\in[-1,1]} |F_n(x)| = 1.$$

Then for all real x, $|x| > 1$ the inequality holds $|F_n(x)| \leqslant |T_n(x)|$.

2.4. Let $P(x,y)$ be a polynomial of the form $\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} a_{ij} x^i y^j$ for some non-negative integers m, n.

a) Let $a_{mn} = 2^{m-1}2^{n-1}$. Prove that $\max\limits_{-1\leqslant x,y\leqslant 1} |P(x,y)| \geqslant 1$. The equality occurs whenever $P(x,y) = T_m(x)T_n(y)$.

b) Let coefficients of $P(x,y)$ be integer and $P(x,y)$ is neither constant in x nor in y. Prove that $\max\limits_{-2\leqslant x,y\leqslant 2} |P(x,y)| \geqslant 4$.

2.5. Let x_k be the roots of Chebyshev's polynomial T_n, where n is even. Prove that $\sum\limits_{k=1}^{n} \frac{1}{x_k^2} = n^2$.

2.6. Prove that for all natural m, n (где $m > n$) and integer x the number

$$(T_{m+n}(x) - 1)(T_{m-n}(x) - 1)$$

is a perfect square.

3 The properties of polynomials with small deviation

3.1. Let $|ax^2 + bx + c| \leqslant 1$ for all $|x| \leqslant 1$. Prove that $a^2 + b^2 + c^2 \leqslant 5$.

3.2. Let $|ax^3 + bx^2 + cx + d| \leqslant 1$ for all $|x| \leqslant 1$. Prove that: $|a| \leqslant 4$, $|a| + |b| \leqslant 4$, $|c| \leqslant 3$, $|a| + |b| + |c| + |d| \leqslant 7$.

3.3. Let $P(x)$ be a polynomial of degree d such that $|P(x)| \leqslant 1$ for all $|x| \leqslant 1$. Prove that

$$|P(2)| < 4^d.$$

3.4. Let $P(x)$ be a polynomial of degree at most 2018 such that for all $x \in [-2, 2]$ we have $|P(x)| \leqslant \frac{1}{|x-\sqrt{3}|}$. Prove that $|P(\sqrt{3})| \leqslant 2019$.

3.5. Let $P(x)$ be a polynomial of degree at most n such that for all $0 < x < 1$ we have $|P(x)| < \frac{1}{\sqrt{x}}$. Prove that $|P(0)| \leqslant 2n + 1$.

3.6. Let $P(x) = a_d x^d + \cdots + a_0$ such that for all $x \in [-1, 1]$ we have $|P(x)| \leqslant 1$. Prove that

$$|a_d| + |a_{d-1}| \leqslant 2^{d-1}.$$

3.7. Let $P(x) = a_n x^n + \cdots + a_0$ be a polynomial of degree at most n with real coefficients such that $|P(x)| \leqslant 1$ where $x \in [-1, 1]$. Let $T_n(x) = t_n x^n + \cdots + t_0$ be n-th Chebyshev polynomial. Prove that

a) $|a_{n-2m}| \leqslant |t_{n-2m}|,\ m = 0, \cdots, \lfloor \frac{n}{2} \rfloor$. The equality occurs whenever $P(x) = \pm T_n(x)$.

b) $|a_{n-2m}| + |a_{n-2m-1}| \leqslant |t_{n-2m}|,\ m = 0, \cdots, \lfloor \frac{n-1}{2} \rfloor$. The equality occurs whenever $P(x) = \pm T_n(x)$.

3.8. (Bernstein's inequality) If $P(x)$ is polynomial with real coefficients of degree n and $|P(x)| \leqslant 1$ on $[-1, 1]$, then $|P'(x)| \leqslant \frac{n}{\sqrt{1-x^2}}$ on $(-1, 1)$.

3.9. (Markov's theorem) Let polynomial $P(x)$ of degree n satisfy the inequality $|P(x)| \leqslant 1$ for $x \in [-1, 1]$. Prove that

$$|P'(x)| \leqslant n^2.$$

The equality is archieved only for polynomials $P = \pm T_n$ and only in the points $x = \pm 1$.

4 Lagrange interpolation formula

A polynomial of degree n is defined uniquely by its values in $n + 1$ points, and the explicit formula may be given. Let us choose (pairwise distinct) points $x_1, x_2, \ldots, x_{n+1}$ and we look for a polynomial $F(x)$ that takes on the values y_k in the points x_k:

$$F(x_k) = y_k, \qquad k = 1, 2, \ldots, n + 1.$$

Set $G(x) = (x - x_1)(x - x_2) \ldots (x - x_{n+1})$. In the product defining $G(x)$, omit the k-th parenthesis and consider the fraction, in which the enumerator contains all the remaining parentheses, and the denominator contains the same parentheses, but in them x_k is substituted instead of x:

$$\Pi_k(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_{n+1})}{(x_k - x_1)(x_k - x_2) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_{n+1})}.$$

It is evident that $\Pi_k(x_k) = 1$ and $\Pi_k(x_j) = 0$ for $j \neq k$. The product $\Pi_k(x)$ may be written in brief form

$$\Pi_k(x) = \frac{G(x)}{G'(x_k)(x - x_k)}.$$

Then the polynomial of interest $F(x)$ is defined by the formula (*Lagrange interpolation formula*):

$$F(x) = \sum_{k=1}^{n+1} y_k \cdot \frac{G(x)}{G'(x_k)(x - x_k)}.$$

This formula may be also used for investigating the coefficients of polynomial F by examining the coefficients of x^k in the both sides of this identity.

4.1. Prove that $\displaystyle\sum_{k=2}^{n-1} \frac{1}{\sin^2 \frac{(k-1)\pi}{n-1}} = \frac{n(n-2)}{3}$.

4.2. Let n be a positive integer and $\theta_k = (k - \frac{1}{2})\frac{\pi}{n}$, $k = 1, \ldots, n$. Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{\sin^2 \frac{\theta_k}{2}} = 4n^2$.

4.3. Let $x_k = \cos\theta_k$, $k = 1, \ldots, n$. Prove that any polynomial $P(x)$ of degree not less than $(n-1)$ satisfies the identity

$$P(x) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}P(x_k)\sqrt{1-x_k^2}\cdot\frac{T_n(x)}{x-x_k}.$$

4.4. Let $x_k = \cos\theta_k$, $k = 1, \ldots, n$. Prove that

$$\sum_{k=1}^{n}(1-xx_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2 = 1.$$

5 Deviation on the other sets

5.1. Let $P(x) = a_d x^d + \cdots + a_0$ be a polynomial with real coefficients such that for all $x \in [0,1] \cup [99,100]$ we have $|P(x)| \leqslant 1$. Find the maximum possible value of $P(50)$, a) if $d = 2$; b) if $d = 100$.

5.2. Let A be the union of a finite number of segments on real line. *Ilya* found fickle polynomial $Q(x)$ with real coefficients with the leading coefficient 1 such that $|Q(x)| < 1{,}999$ for all $x \in A$. Prove that *Navid* may find fickle polynomial $P(x)$ with real coefficients and the leading coefficient 1 such that $|P(x)| < 1$ for all $x \in A$.

5.3. Three non-intersecting segments $[-b; -a]$, $[-c; c]$ and $[a; b]$ are given on the real line, and $b^2 = a^2 + c^2$. Prove that polynomial $f(x)$ of degree $2n$ with the leading coefficient 1, cannot be less than $2\left(\frac{ac}{2}\right)^n$ by the absolute value on these segments.

5.4. Pólya's theorem. Let $S \subset \mathbb{R}$ be a union of finite number of non-intersecting segments, let ℓ be the sum of lengths of these segments. Then for any polynomial $f(x)$ of degree n with the leading coefficient 1 there exists number $y \in S$ such that

$$|f(y)| \geqslant 2\left(\frac{\ell}{4}\right)^n.$$

And if S is not a single segment, then the sign in the inequality is strict.

5.5. Find the polynomial of degree n that has the minimum deviation on the set $\{0, 1, 2, \ldots, n\}$.