Schiffler point
Solutions

Ivan Kukharchuk, Leonid Shatunov, Konstantin Shcherbakov

Section 1. The radical center of Euler circles

Here and below we consider the triangle ABC. Points A_1, B_1, C_1 are the midpoints of the arcs BC, AC and AB of the circumcircle Ω of the triangle ABC, which don't contain points A, B and C respectively, I is the center of the incircle of this triangle, and O is the center of Ω. Points M_a, M_b and M_c are the midpoints of BC, AC and AB. The incircle of $\triangle ABC$ touches sides BC, AC and AB at points K_a, K_b and K_c.

Points A_2, B_2, C_2 are symmetric to the points A_1, B_1, C_1 with respect to the sides BC, AC and AB respectively. The points E_a, E_b and E_c are the midpoints of the segments IA_2, IB_2 and IC_2, E is the midpoint of IO.

Ex. 1. Prove that E is the center of the nine-point circle $\triangle A_1B_1C_1$

Solution. I is orthocenter and O is the circumcenter, respectively.

Ex. 2. Prove that:
(a) $\triangle A_1OI \sim \triangle A_1IA_2$.
(b) $\angle C_1A_1E = \angle E_aA_1B_1$.
(c) The lines A_1E_a, B_1E_b, C_1E_c intersect at one point.

Solution. **(a)** $\angle A_1A_2B = \angle A_2A_1B = \angle OBA_1 \Rightarrow A_1O \cdot A_1A_2 = A_1B^2 = A_1I^2$.
(b) From the similarity proved in a) A_1E and A_1E_a are isogonal with respect to the angle $\angle IA_1O$. In addition, A_1O and A_1I are isogonal with respect to angle $\angle C_1A_1B_1$ as directions to the circumcenter and the orthocenter. We obtain the required.
(c) This point is isogonally conjugate to point E with respect to triangle $A_1B_1C_1$.

We denote by H_c, H_a and H_b the orthocenters of triangles AIB, BIC and CIA respectively.

Ex. 3. **(a)** Prove that A_2 is the circumcenter of BH_aC.
(b) Prove that E_a is the center of the nine point circle of a triangle BIC.

Solution. **(a)** Consider symmetry with respect to BC, then the circle BH_aC maps to the circle BIC, and A_2 maps to its center A_1.
(b) A_2 and I are the circumcenter and the orthocenter of $\triangle BH_aC$ respectively. Then E_a is the center of the circle of the nine points $\triangle BIC$, and is congruent to $\triangle BH_aC$.

Problem 1 (Schiffler Point)**.** Prove that the Euler lines of triangles AIB, BIC and CIA intersect at a single point.

Solution. Corollary of Exercises 2c and 3b.

We denote the Schiffler point of triangle ABC by Sh.

Let ω_a be the excircle of triangle ABC, touching side BC at point T and the extensions of sides AB and AC at points L and N.The circles ω_b, ω_c are defined similarly. Points I_a, I_b and I_c are the centers of ω_a, ω_b and ω_c respectively.

Ex. 4. Let I' be the center of the incircle of triangle $M_aM_bM_c$. Prove that the midpoint of segment AA_2 is the center of the circle $(M_bI'M_c)$.

Solution. Consider homothety with ratio 2 and center A followed by point reflection at point M_a (this transform coincides with homothety with center at the point of intersection of the medians with ratio -2). Then the triangle $M_aM_bM_c$ maps to the triangle ABC, so I' maps to I. It remains to note that the midpoint of AA_2 maps to A_1, which is the center of the circle BIC.

Ex. 5. Prove that the Euler line of triangle $M_bI'M_c$ is parallel to line A_2I_a.

Solution. By the same transformation, the Euler line $M_bI'M_c$ maps to the Euler line of triangle BIC, that is, to A_1E_a. But this is the midline in triangle II_aA_2, so both Euler lines are parallel to A_2I_a.

Ex. 6. The circle α_1 is the image of a circle (I_bCA) in symmetry with respect to I_bC, circle α_2 is the image of a circle (I_cBA) with symmetry with respect to I_cB. Prove that the point A_2 lies on the radical axis of circles α_1 and α_2.

Solution. Note that I belongs to circumcircles of triangles AI_cB и AI_bC. We denote by K_b and K_c the reflections of I with respect to I_bC and I_cB respectively, so $K_b \in \alpha_1$ and $K_c \in \alpha_2$, K_bI_b is the diameter of α_1 and K_cI_c is the diameter of α_2. Since $I_bC \perp IC$ and $I_cB \perp IB$, we have that B is the middle of segment IK_c and C is the middle of IK_b. Note also that point I_a belongs to the radical axis of circles α_1 and α_2, so quadrilateral I_cBCI_b is inscribed and thus $I_aB \cdot I_aI_c = I_aC \cdot I_aI_b$.

Consider homothety with center I and ratio $\frac{1}{2}$. Point A_2 maps to E_a, point K_b maps to C, K_c maps to B, I_a maps to A_1, I_b maps to B_1, I_c maps to C_1. Circles α_1 and α_2 map to circles with diameters B_1C and C_1B respectively (we denote them by β_1 and β_2).

Note that H_a belongs to the radical axis of circles β_1 and β_2: using notation $X = H_aB \cap CI$, $Y = H_aC \cap BI$ we have that $X \in \beta_2$, $Y \in \beta_1$ and since $H_aB \perp CI$, $H_aC \perp BI$ quadrilateral $BXYC$ is inscribed therefore $H_aX \cdot H_aB = H_aY \cdot H_aC$.

Hence we obtain that H_aA_1 is the radical axis of circles β_1 and β_2. $E_a \in H_aA_1$ (see exercise 3) and this finishes the proof.

Problem 2 (J.-P. Ehrmann, P. Yiu, K. L. Nguyen)**.** Prove that the radical center of the Euler circles of triangles BI_aC, CI_bA and AI_cB is the Schiffler point for triangle $M_aM_bM_c$.

Solution. $I_a = AI \cap CI_b \cap BI_c \Rightarrow \text{pow}_{\alpha_1}(I_a) = \text{pow}_{(CI_bA)}(I_a) = \text{pow}_{(AI_cB)}(I_a) = \text{pow}_{\alpha_2}(I_a) \Rightarrow A_2I_a$ is the radical axis of α_1 and α_2.

Let us make a homothety with center in A and coefficient $\frac{1}{2}$. Then in the conditions of Eq. 6, the images α_1 and α_2 are nine point circles $\triangle CI_bA$ and $\triangle AI_cB$. Hence, the image A_2I_a lies on the radical axis of the images α_1 and α_2 and by Exercises 4 and 5 is the Euler line $\triangle M_bI'M_c$.

Consider a triangle Δ formed by lines similar LN (i.e. connecting the points of tangency of the excircles with the extensions of the corresponding sides) and triangle Θ with vertices at the midpoints of arcs M_bM_c, M_cM_a and M_aM_b of the circumcircle of the triangle $M_aM_bM_c$.

Ex. 7. Prove that triangle Δ is homothetic to triangle Θ

Solution. Making homothety in M with coefficient -2 triangle Θ passes into triangle $A_1B_1C_1$, hence that the sides of both triangles are perpendicular to the corresponding bisectors of triangle ABC (see Eq. 10a).

Ex. 8. Prove that the circumcenter Δ is the orthocenter of the original triangle ABC, and the circumcenter Θ is the center of the Euler circle $\triangle ABC$.

Solution. The second statement is true due to the construction of Θ. To prove the first statement we formulate the following lemma:

Lemma 1. *Given circles ω_1 and ω_2 with drawn common outer tangent AB and two inner tangents CD and EF ($A, C, E \in \omega_1$; $B, D, F \in \omega_2$). Lines CD and EF intersect at a point P, and lines AC and BF intersect at a point X. Prove that $PX \perp AB$.*

Proof. Let Q be the point of intersection of outer tangents to ω_1 and ω_2, and Y be the point of intersection of AE and BD. $AC \perp BD$, since the first line is perpendicular to the bisector of the angle between AB and CD, and the second line is parallel. Similarly $AE \perp BF$. Let Γ be a circle with diameter AB. Then $\Gamma \perp \omega_1$ and $\omega_2 \Rightarrow P$ lies on the polar Q with respect to Γ, i.e., on the line XY. Note that A, B, X, Y orthocentric quadrilateral. We get the required one. □

Let us continue the solution of Problem 8. Let us denote the vertices of triangle Δ respectively A_d, B_d, C_d. By Lemma 1 $AA_d \perp BC$. Let the bisector $\angle ACB$ intersects B_dA_d at the point H_d, H_a is the base of the altitude from vertex A of triangle ABC. Then the quadrilateral $A_dH_dH_aC$ is inscribed and $\angle B_dA_dA = \frac{\angle C}{2} \Rightarrow A_dA$ is the direction to the circumcenter in $\triangle A_dB_dC_d$. We get what we need.

Ex. 9. Prove that the vertices of triangle Δ lie on the radical axes of the corresponding pairs of Euler circles of triangles BI_aC, CI_bA and AI_cB.

First we prove a lemma:

Lemma 2. *Let AH_a, BH_b, CH_c be the altitudes of the triangle ABC. Points P and Q are the reflections of point H_a through the sides AB and AC respectively. Prove that points P, Q, H_b, H_c are collinear.*

Proof. Set $R := BP \cap CQ, X := BP \cap CH_c, Y := CQ \cap BH_b, H$ is orthocenter $\triangle ABC$. X is the image of C, and H is the image of the orthocenter $\triangle ABX$ with symmetry relative to AB. Hence, $H \in (XAB)$, similar to $H \in (YAC) \Rightarrow A$ The Miquel point of the $BHCR$ quadrilateral. The projections of the Miquel point on the sides of the quadrangle lie on the same straight line (proved by consistent application of Simson's theorem). We get the required □

Lemma 3. *Let AA_1, BB_1, CC_1 be the altitudes of the triangle ABC and let A_1A_b and A_1A_b be perpendiculars from point A_1 to AB and AC respectively. Points B_a, B_c, C_a, C_b are defined similarly. Prove that points A_b, A_c, B_a, B_c, C_a, C_b are concyclic.*

Proof. It's a known fact: A_bA_c and BC are antiparallel with respect to $\angle A$. Also B_cC_b and H_bH_c are antiparallel, i.e. B_cC_b and BC are parallel. So $A_bA_cB_cC_b$ is an inscribed quadrilateral. $A_cC_a \parallel AC$ and $B_cC_b \parallel BC$. Since C_aC_b and AB are antiparallel with respect to $\angle C$, the same is true for the angle between the lines A_cC_a and $B_cC_b \Rightarrow$ quadrilateral $A_cB_cC_aC_b$ is inscribed. Hence we get that pentagon $A_cB_cC_aC_bA_b$ is inscribed and similarly we get that the whole hexagon is inscribed. □

Now we return to the solution of exercise 9. Using Lemma 2 for $\triangle ACI_b$ we obtain that the line A_dC_d contains feet of altitudes of this triangle falling on its sides AI_b and CI_b. Similarly the line A_dB_d contains feet of altitudes of $\triangle ABI_c$ falling on its sides AI_c and BI_c. These 4 feet of altitudes are concyclic by Lemma 3 applied to $\triangle I_aI_bI_c$. This finishes the proof.

Problem 3. Prove that the radical center of the Euler circles of triangles BI_aC, CI_bA и AI_cB lies on the Euler line of triangle ABC.

Solution. Consider a homothety h mapping the triangle δ to the triangle θ. By exercise 8, h maps the orthocenter of $\triangle ABC$ to the centre of the nine-point circle of $\triangle ABC$. The radical axis of circles (I_bAC) and (I_cAB) maps to the direction to the Schiffler point of triangle $M_aM_bM_c$ from the middle of the arc M_bM_c. Thus according to Problem 2 we have that the Schiffler point of $\triangle M_aM_bM_c$ is fixed under this homothety, therefore it's the homothetic center. Hence it belongs to the line containing the centre of the nine-point circle and the orthocente of $\triangle ABC$, i.e. to the Euler line.