Schiffler point

Ivan Kukharchuk, Leonid Shatunov, Konstantin Shcherbakov

The project is presented by:
I. Kukharchuk, L. Shatunov, P. Kozhevnikov, V. Oganisyan, A. Matveeva

And it takes billions of years for stars to be born
We don't have time to wait for them
So I draw between the red dots
Hundreds of lines to create new constellations
They lit up everything and I could see where to run

pyrokinesis

Introduction

This project is intended for fans of classical planimetry. We will study the properties of the mysterious Schiffler point Sh. It is defined as follows: let I be the incenter of an arbitrary triangle ABC, then the Euler lines of triangles AIB, BIC, AIC and ABC intersect at this point Sh.

In our project, during the first two parts we will use only «synthetic methods» (without using trigonometry, cartesian coordinate method or any other calculation, cubic curve techniques, or knowledge from projective geometry).

With such a limited arsenal of methods, we will be able to obtain a whole series of brilliant results: find the location of the Schiffler point with the respect to other triangle centers lying on the Euler line of triangle ABC; catch directions to the Schiffler point from the vertices of triangle ABC; connect the Schiffler point to the radical center of the Nine-poin circle of triangles like BI_aC (where I_a — is the center of the excircle of triangle ABC, wich touches segment BC).

Part 3 of the project contains facts about Schiffler external points. We call a Schiffler external point Sh_a the point that arises if we replace the point I with the point I_a in the definition of a point Sh everywhere. Part 3 is suggested to the readers not for solving, but only for familiarization, because the proofs of the facts about point Sh_a are absolutely similar to the proofs of the facts about point Sh. The facts from this part are not evaluated as part of the project at the conference.

When we are looking at the construction that we will use to find directions from the vertices of triangle ABC to the Schiffler point, the question about its geometric nature immediately arises — its essence is so unobvious. The answer to this question will be given in Part 4.

It turns out that the Schiffler point is the pole of the line IO (where O — is the circumcenter of triangle ABC), and the construction in question is just a straightedge construction.

However, the pole here is considered with the respect to some hyperbola. So in Part 4 we give all the necessary theory for working with conics as part of our project. It is assumed that the reader, who work with conics for the first time, will have no problems in solving tasks of this part and will not need any additional knowledge to work in the project.

It turns out that almost all proofs of problems in Part 4 have projective origin. Therefore, in Part 5 we give a beautiful projective generalization of the Schiffler point in which the generalized «incenter» I and the generalized «excenters» I_a, I_b and I_c play the same role, and hence the Schiffler point together with the external Schiffler points become equal in this generalization.

Part 6 is dedicated to another generalization of the Schiffler point existence problem: find the locus of all points P of the plane for which the Euler lines of triangles APB, BPC, APC intersect at the same point. It turns out that this is a rather complicated object - we can look at it as a curve of degree six on the projective plane $\mathbb{C}\mathrm{P}^2$, but it can be interpreted the product of a cubic, an infinitely distant line, and the circumcircle of the original triangle.

The cubic in question was named by the Luxembourger Joseph Neuberg and in the «Encyclopedia of the triangle cubics» [6] is on the first place! We will a bit study this object and get different alternative beautiful descriptions. More properties of the Neuberg cubic can be found in the SCTT-2017 project [8].

The fact that a Schiffler point exists was proposed as a problem by the American engineer, businessman, and amateur geometer Kurt Schiffler in the Feral issue of Crux Mathematicorum [1] in 1985. And Schiffler point became one of the most brilliant and new discoveries in triangle planimetry. In the «Encyclopedia of Triangle Centers» [2] Schiffler point is on the honorable 21st place.

The first existence proof was given by the Dutch G.R. Veldkamp and W.A. van der Spek in the June 1986 issue of journal Crux Mathematicorum [3] (i.e., the problem had been unsolved for more than a year!). As noted in [5], an independently similar geometric proof I.F. Shariguin came up with, and he is one of the most famous modern set compositors and geometers.

A great contribution to the study of the Schiffler point was made by our contemporary Russian geometer L. A. Emelyanov ([4] and [5]). The authors would like to thank L. A. Emelyanov for his inspiring to create the project.

We wish success to the conference participants in solving the problems!

Part 1. The radical center of Euler circles

Here and below the triangle is given ABC. The points A_1, B_1, C_1 — are the midpoints of the arcs BC, AC and AB, not containing points A, B and C respectively, of *the circle Ω circumcircle about triangle ABC, I — is the center of the incircle of this triangle, and O — is the center of Ω. Points M_a, M_b and M_c — are the midpoints of BC, AC and AB. The incircle $\triangle ABC$ touches sides BC, AC and AB at points K_a, K_b and K_c.*

Points A_2, B_2, C_2 are symmetric to the points A_1, B_1, C_1 with respect to the sides BC, AC and AB respectively. The points E_a, E_b and E_c — are the midpoints of the segments IA_2, IB_2 and IC_2, E — is the midpoint of IO.

Ex. 1. Prove that E — is the center of the nine point circle $\triangle A_1B_1C_1$

Ex. 2. Prove that:
(a) $\triangle A_1OI \sim \triangle A_1IA_2$.
(b) Prove that $\angle C_1A_1E = \angle E_aA_1B_1$.
(c) Prove that the lines A_1E_a, B_1E_b, C_1E_c intersect at one point.

Let us define the points H_c, H_a and H_b — are orthocenters of triangles AIB, BIC and CIA respectively.

Ex. 3. **(a)** Prove that A_2 — is the center of the circumcircle of BH_aC.
(b) Prove that E_a is the center of the nine point circle of a triangle BIC.

Problem 1 (Schiffler Point)**.** Prove that the Euler lines of triangles AIB, BIC and CIA intersect at a single point.

We denote the Schiffler point of triangle ABC by Sh.

We have not yet proved that the Euler line of triangle Sh passes through the point ABC. Let us postpone this and prove an interesting fact, but before we do so, we must again overcome a few auxiliary statements.

Let ω_a — be the excircle of triangle ABC, tangent the side BC at the point T and the extensions of the sides AB and AC at the points L and N. Similarly the circles ω_b, ω_c. Points I_a, I_b and I_c — are the centers of ω_a, ω_b and ω_c respectively.

Ex. 4. let I' — be the center of the incircle of triangle $M_aM_bM_c$. Prove that the midpoint of segment AA_2 — is the center of the circle $(M_bI'M_c)$.

Ex. 5. Prove that the Euler line of triangle $M_bI'M_c$ is parallel to line A_2I_a.

Ex. 6. The circle α_1 — is the image of a circle (I_bCA) in symmetry with respect to I_bC, circle α_2 — is the image of a circle (I_cBA) with symmetry with respect to I_cB. Prove that the point A_2 lies on the radical axis of circles α_1 and α_2.

Problem 2 (J.-P. Ehrmann, P. Yiu, K. L. Nguyen)**.** Prove that the radical center of the Euler circles of triangles BI_aC, CI_bA and AI_cB is the Schiffler point for triangle $M_aM_bM_c$.

In [7] other curious properties of the radical axis of Euler circles of triangles CI_bA and AI_cB. In particular, it is claimed that the outer Feuerbach point F_a of triangle ABClies on it. Interested readers can try to prove this fact on their own or figure out its solution in the source.

Consider a triangle Δ formed by lines similar LN (i.e. connecting the points of tangency of the excircles with the extensions of the corresponding sides) and triangle Θ with vertices at the midpoints of arcs M_bM_c, M_cM_a and M_aM_b of the circumcircle of the triangle $M_aM_bM_c$.

Ex. 7. Prove that triangle Δ is homothetic to triangle Θ.

Ex. 8. Prove that the center of the circumcircle Δ coincides with the orthocenter of the original triangle ABC, and the center of the circumcircle Θ coincides with the center of the Euler circle ABC.

Ex. 9. Prove that the vertices of triangle Δ lie on the radical axes of the corresponding pairs of Euler circles of triangles BI_aC, CI_bA and AI_cB.

Problem 3. Prove that the radical center of the Euler circles of triangles BI_aC, CI_bA и AI_cB lies on the Euler line of triangle ABC.

In particular, it follows that the Schiffler point also lies on the Euler line of triangle ABC. However, there is a simpler way to prove this, which we will we will discuss in the next section.

Part 2. Direction to Schiffler point

Let's recall one useful statement that will often help below.

Proposition 1. *Two unequal triangles are homothetic if and only if their corresponding sides are parallel.*

Here is one of the most famous applications of this fact.

Ex. 10. Prove that:
(a) $A_1C_1 \perp BI$
(b) triangles $A_1B_1C_1$ и $K_aK_bK_c$ are homothetic;
(c) point I lies on the Euler lines $A_1B_1C_1$ and $K_aK_bK_c$;
(d) A_1K_a, B_1K_b, C_1K_c and IO intersect at one point.

In a similar way, we can prove the following "close relative" of the previous statement, which will come in handy later.

Ex. 11. Prove that:
(a) O — is the center of the nine point circle of triangle I_bII_c;
(b) I_aO is the Euler line of triangles LTN and I_bII_c.

Let us now proceed to the proof of the key theorem of this part.

Ex. 12. Point X and Y are such that $I \in XY$, $XA \parallel IB$, $YA \parallel IC$ and $XY \parallel BC$. Prove that:

(a) Triangles XAY and BIC are homothetic;

(b) Triangles XAY and BIC have the same circumcenters;

(c) The line connecting the intersection of the medians $\triangle XAY$ and $\triangle BIC$, contains the center of the circumcircle $\triangle XAY$.

Let M be the point of intersection of the medians, r be the radius of the incircle, R is the radius of the circumcircle of triangle ABC.

Ex. 13. Let M_1 be the point of intersection of the medians of triangle XAY. Prove that $MM_1 = \frac{2}{3}r$.

The problem below is a consequence of the statements just proved. It gives us information about the ratio in which the Schiffler point divides the segment MO and, at the same time, is an alternative way of proving the existence of the Schiffler point.

Problem 4 (G.R. Veldcamp, V.A. van der Spek, I. F. Sharygin)**.** Prove that the Euler line of triangle BIC divides segment MO by $\frac{2r}{3R}$, counting from the point M.

Here is a useful simple consequence of this fact, which will help us to prove a number of properties of the Schiffler point.

Let ASh intersect the line OM_a at the point Z_a.

Ex. 14. Prove that $\dfrac{M_aZ_a}{Z_aO} = \dfrac{r}{R}$.

Let $U \neq L$ a point on the line LN such that $BL = BU$. Let $V \neq N$ a point on the line LN such that $CN = CV$.

Ex. 15. Prove that $\triangle UTV$ и $\triangle K_bK_aK_c$ are symmetric with respect to M_a.

Ex. 16. Prove that $\triangle I_aI_bI_c$ is similar to $\triangle UTV$ with factor $\dfrac{2R}{r}$.

Let point P_a be the symmetry of point T with respect to LN. The points P_b and P_c are defined similarly. Let W be the intersection of lines I_aT b AP_a.

Let point Q_a e the point of intersection of lines OI_a (by Ex. 11b, this is the Euler line LTN) u BC. The points Q_b and Q_c are defined similarly.

Ex. 17. Prove that $\dfrac{TW}{WI_a} = \dfrac{r}{R}$ *(hint: use the previous exercise).*

As we can see, we have obtained a relation similar to the one in Ex. 14. We will use this a little later, but for now we will prove another auxiliary result.

Let D be the intersection of AI_a and BC. Mid perpendicular to AD intersects the lines BI_a u CI_a at the points G and F.

Ex. 18. Prove that triangle DGF s homothetic to triangle TNL and the center of their homothety lies at the intersection of lines BC and AP_a.

Ex. 19. Prove that AP_a passes through Q_a.

Problem 5 (L. A. Emelyanov и T. L. Emelyanova)**.** Prove that

- AQ_a, BQ_b and CQ_c intersect at the Schiffler point of the triangle ABC;
- AP_a, BP_b and CP_c intersect at the Schiffler point of the triangle ABC.

Let H' be the orthocenter of triangle $K_aK_bK_c$.

Ex. 20. Prove that the quadrilaterals K_cAK_bH' and $NALP_a$ are similar.

Problem 6. Prove that the Schiffler point is isogonally conjugate with respect to triangle ABC to the point H'.

Let's talk now about the analog of the Schiffler point. As we know,

Part 3. «Outside» Schiffler points

Let's now talk about analogues of the Schiffler point. Let's replace the incenter with the eccenter in the problems proposed above and carry out similar reasoning. The facts presented in this part are not necessary for the decision, will not be evaluated and are given for informational purposes only.

Fact 1 («Outside» Schiffler point)**.** Let I_a — be the center of an excircle $\triangle ABC$, tangent to side BC. Prove that the Euler lines of triangles ABC, AI_aB, AI_aC, BI_aC have a common point, which divides segment MO externally with respect to $\frac{2r_a}{3R}$, counting from the point M.

We denote the outer $A-$ Schiffler point by Sh_a. Similarly, we define Sh_b and Sh_c.

Fact 2. Symmetric point of K_a with respect to K_bK_c lies on ASh_a.

Fact 3. Intersection of IO and BC lies on ASh_a.

Fact 4. BSh_a, I_cO and AC intersect at the same point.

Let A_0, B_0, C_0 — be the midpoints of the large arcs BC, CA, AB of the circumcircle of triangle ABC respectively.

Fact 5. Sh_a is isogonally conjugate to the midpoint of OI_a with respect to triangle $A_1B_0C_0$.

Fact 6. The radical center of the Euler circles ACI_c, ABI_b and BIC is a M_a- Schiffler point $\triangle M_aM_bM_c$

Part 4. References

[1] K. Schiffler (1985) Problem 1018 https://cms.math.ca/wp-content/uploads/crux-pdfs/Crux_v11n02_Feb.pdf *Crux Mathematicorum* 11: 51. Retrieved September 24, 2023.

[2] C. Kimberling X0021: Schiffler point https://faculty.evansville.edu/ck6/encyclopedia/ETC.html *Triangle centers and central triangles*

[3] G. R. Veldkamp, W. A. van der Spek (1986) Solution to Problem 1018. https://cms.math.ca/wp-content/uploads/crux-pdfs/Crux_v12n06_Jun.pdf *Crux Mathematicorum* 12: 150–152. Retrieved September 24, 2023.

[4] L.Emelyanov and T. Emelyanova (2003) A Note on the Schiffler Point. https://forumgeom.fau.edu/FG2003volume3/FG200312.pdf *Forum Geometricorum* Vol. 3, 113-116.

[5] L.Emelyanov (2006) The Schiffler Point. In memory of I.F. Sharygin. https://geometry.ru/persons/emelyanov/shiffler.pdf *Mathematics at school* № 6, 58 - 60.

[6] B. Gilbert K001: Neuberg cubic http://bernard-gibert.fr/Exemples/k001.html *Cubics in the Triangle Plane.*

[7] K. Nguyen (2005) On the Complement of the Schiffler Point. https://forumgeom.fau.edu/FG2005volume5/FG200521.pdf *Forum Geometricorum* 5: 149–164.

[8] N. Belukhov, A. Zaslavskii (2017) Fermat points, Euler lines and something else. https://turgor.ru/lktg/2017/2/2-1ru.pdf *The Summer Conferences of the Tournament of Towns* 2017.

[9] F. Bakharev (2021) A conversation about isosceles (equilateral, rectangular) hyperbolas. https://www.youtube.com/watch?v=xapJu-mT86c

[10] A. Akopyan, A. Zaslavskii (2007) Geometric properties of quadratic curves. https://old.mccme.ru/free-books/akopyan/Zaslavky-Akopyan.pdf *MCCME Publishing*

[11] D. Brodskii (2022) Motion of points. https://turgor.ru/lktg/2022/4/4-1-moving-rus.pdf *The Summer Conferences of the Tournament of Towns* 2022.