Point moving

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Life requires movement

Aristotle

1 Projective movement

Lemmas and theorems formulated in this section may be used without proof. As usually they are standard assertions of projective geometry — if you want it is not hard to verify their correctness.

Definition 1.1 Projective plane. *The projective plane $\mathbb{RP}^2$ is the Euclidean plane completed by infinite points, each such point is the common point of some class of parallel lines. So each line is completed by one infinite point, and this point is the same for all parallel lines. All infinite points form the infinite line. A line completed by an infinite point is called a projective line.*

Definition 1.2 Cross-ratio of four points on a projective line *Let points A, B, C, D lie on a projective line ℓ. The cross-ratio $(A, B; C, D)$ of these points on ℓ is the value*

$$\frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}}$$

We suppose that an infinite point divide any segment in ratio $1 : 1$.

Definition 1.3 Pencil of lines. *The pencil of lines $\mathcal{L}_A$ of point A is the set of all lines passing through A.*

Not that the set of all parallel lines is the pencil of the corresponding infinite point.

Definition 1.4 Cross-ratio of four lines on a pencil. *Let lines a, b, c, d pass through a point O (the set of lines passing through O is a pencil). Choose arbitrary directing vectors $\overrightarrow{v}_a, \overrightarrow{v}_b, \overrightarrow{v}_c, \overrightarrow{v}_d$ on these lines. The cross-ratio $(a, b; c, d)$ of the given lines is the value*

$$\frac{\sin\angle(\overrightarrow{v}_c, \overrightarrow{v}_a)}{\sin\angle(\overrightarrow{v}_c, \overrightarrow{v}_b)} : \frac{\sin\angle(\overrightarrow{v}_d, \overrightarrow{v}_a)}{\sin\angle(\overrightarrow{v}_d, \overrightarrow{v}_b)}$$

Remark: *Generally the sign of $\sin\angle(\overrightarrow{v}_x, \overrightarrow{v}_y)$ is not defined by the lines x and y, because we may choose the directing vector by two ways. But if we change the direction of one vector, the signs of two sines invert, and the value of the cross-ratio does not change.*

Definition 1.5 Cross-ratio of points on a circle *Let points A, B, C, D lie on a circle Ω. Mark an arbitrary point O* **on this circle**. *The cross-ratio $(A, B; C, D)$ of points on Ω is he cross-ratio of lines $(OA, OB; OC, OD)$. This definition is correct because the value of an inscribed angle is constant.*

Lemma 1.1 *Let points A, B, C, D lie on a line ℓ, and a point O lie not on this line. Then*

$$(A, B; C, D) = (OA, OB; OC, OD)$$

1. Let we consider points of the plane as complex numbers. Make sure that the cross-ratio of four points of a line or a circle corresponding to a, b, c, d equals $\frac{c-a}{c-b} : \frac{d-a}{d-b}$.

Later we will define the cross-ratio for four points of an arbitrary conic. In this case the result of the last problem **IS NOT CORRECT**.

Definition 1.6 Homography. *A map $\mathcal{F}$ is called projective, or homography if*

$$(\mathcal{F}(A), \mathcal{F}(B); \mathcal{F}(C), \mathcal{F}(D)) = (A, B; C, D)$$

. The map may transform a line, a pencil or a circle (a conic) to a line, a pencil or a circle (a conic).

Lemma 1.2 *Let $A, B, C \in \ell, k \in \mathbb{R}$, then there exists a unique point $D \in \ell$ such that $(A, B, C, D) = k$.*

Theorem 1.1 *Each homography of lines may be presented as a composition of several central projections and parallel translations.*

Definition 1.7 Projective movement. *Fixe some projective line $\mathcal{T}$. Say that a point X moves on a projective line or a circle ℓ projectively, if there exists a projective map $\mathcal{F}\colon \mathcal{T} \to \ell$ such that $X = \mathcal{F}(t \in \mathcal{T})$. The line $\mathcal{T}$ may be considered as the axis of time.*

A projective rotation of a line on a pencil is defined similarly.

Definition 1.8 The line rotating projectively *The line ℓ rotates projectively in the pencil $\mathcal{L}$ if there exists a projective map $\mathcal{F}\colon \mathcal{T} \to \mathcal{L}$*

such that $\ell = \mathcal{F}(t \in \mathcal{T})$.

Later we will use the following notations for several special homographies.

- $\ell \xrightarrow{S} \mathcal{L}_S$ — projection centered at point S, mapping a line ℓ to a pencil $\mathcal{L}_S$.
- $\mathcal{R}_S^{\psi}$ — rotation centered at S to angle ψ counterclockwise.

- $\mathcal{H}_S^k$ — homothety centered at S with coefficient k

Demonstrate, how the projective movement helps to solve problems.

Example 1.1 *Let A_1 be the touching point of the incircle of triangle ABC with the side BC, and D be an arbitrary point on BC. Denote as I_B and I_C the incenters of triangles $\Delta ABD, \Delta ACD$ respectively. Prove that $\angle I_B A_1 I_C = 90°$*

Divide the solution into several typical steps.

1. Fix ΔABC and move projectively I_B on the bisector of $\angle ABC$.
2. Construct a homography $\mathcal{F}\colon BI \to BI$, the identity of this homography is equivalent to the required assumption.

   Present «a chain» of homographies as a table, denote as ℓ_b, ℓ_c the bisectors of $\angle B$ and $\angle C$ perspectively:

   $$\begin{array}{ccccccccccccc} \ell_b & \xrightarrow{A} & \mathcal{L}_A & \xrightarrow{\mathcal{R}_A^{\frac{\alpha}{2}}} & \mathcal{L}_A & \xrightarrow{A} & \ell_b & \xrightarrow{A_1} & \mathcal{L}_{A_1} & \xrightarrow{\mathcal{R}_{A_1}^{\frac{\pi}{2}}} & \mathcal{L}_{A_1} & \xrightarrow{A_1} & \ell_b \\ I_B & & AI_B & & AI_C & & I_C & & A_1 I_C & & A_1 I'_B & & I'_B \end{array}$$

   We indicate the lines and the pencils transformed by the homographies in the top row, and the corresponding moving objects in the bottom one. The assumption of the problem is equivalent to the identity $I = I'$
3. Verify that the assumption is correct for some three positions of I_B. As usually it is useful to choose degenerated positions. For the given problem consider $I_B = B, I_B = I, I_B = P$, where P is the incenter of triangle ABA_1. For each of these cases the problem is clear. Since a homography is uniquely defined by the images of three points, the constructed homography of ℓ_1 is the identity, hence the assumption is correct for any position of I_B.

   Now train to solve problems using the projective movement. In several problems it is useful to remember that directions are points of the infinite line.

2. A line ℓ rotates projectively around a fixed point P. A point $S \neq P$ is fixed. Prove that the projection of S to ℓ moves projectively. Find the trajectory of this point

3. The external bisectors BB_1 and CC_1 of triangle ABC meet at point I_A. A line ℓ passing through I_A meets AB and AC at points X and Y respectively. Prove that the reflections of BY and CX about BB_1 and CC_1 respectively meet on B_1C_1.

4. Let points B_1, C_1 lie on the sides AC, AB of triangle ABC in such a way that lines BB_1, CC_1 meet on the altitude AA_1. Prove that the lines A_1B_1 and A_1C_1 are symmetric with respect to AA_1.

5. Points A_2, B_2 and C_2 lie on cevians AA_1, BB_1, CC_1 (i.e. concurrent lines) respectively of triangle ABC. Let $A_3 = BC_2 \cap B_2C$. points B_3 and C_3 are defined similarly. Prove that AA_3, BB_3 and CC_3 concur.

6. Let $ABCD$ be a rhombus with acute angle B. Let O the circumcenter of triangle ABC. A point P lie on the extension of OC beyond C. The line PD meets the line passing through O and parallel to AB at point Q. prove that $\angle AQO = \angle PBC$.

7. Let ABC be an acute-angled triangle with circumcircle Ω and incircle ω. A point P lies on a segment joining the centers of Ω and ω. Denote as A', B' and C' the second common points of AP, BP and CP with Ω. Prove that the internal bisectors of $\angle BA'C, \angle CB'A$ and $\angle AC'B$ concur at a point lying on the center line of Ω and ω.

8. Let $ABCD$ be a circumscribed quadrilateral with incenter I. Points P and Q lie on AI and CI respectively in such a way that $\angle PBQ = \frac{1}{2}\angle ABC$. Prove that $\angle PCQ = \frac{1}{2}\angle ADC$

9. Let S be the projection of the orthocenter of triangle ABC to the median AM. A circle ω passes through A and S, and meets AB and AC at Q and P respectively. Prove that BP and CQ meet on ω.

10. a) Let f be a homography of a projective line ℓ to itself. Parametrize the finite points of the line by x. Prove that f is fractionally linear, i.e. $f(x) = \dfrac{ax+b}{cx+d}$, where a, b, c and d are fixed numbers. b) Let f be a homography from ℓ_1 to ℓ_2. Set Cartesian coordinates on a plane. Prove that $f(x,y) = (\dfrac{a_1x+b_1y+c_1}{dx+ey+f}, \dfrac{a_2x+b_2y+c_2}{dx+ey+f})$, where $a_1, a_2, b_1, b_2, c_1, c_2, d, e$ are fixed numbers. c) Prove that any map of $\mathbb{R}^{\not\models}$ defined by the formulas of p. b) may be uniquely extended to a projective map of the projective plane.

11. A map f from a complex line ℓ_1 to itself is defined as $f(z) = \dfrac{P(z)}{Q(z)}$, where P, Q are polynomials. Let f be a bijection. Prove that f conserve the cross-ratio of any four points.

2 Projective movement +

It is a wonder that an inversion limited to a line or a circle also conserve cross-ratios!

Theorem 2.1 *Let an inversion $\mathcal{I}$ maps a circle or a line Ω to a circle or a line $\widetilde{\Omega}$. Then for any four points $A, B, C, D \in \Omega$ we have $(A, B; C, D) = (\mathcal{I}(A), \mathcal{I}(B); \mathcal{I}(C), \mathcal{I}(D))$*

The simplest proof of this theorem use the following properties of complex numbers: a number equals to its conjugated if and only if it is real; the cross-ratio of four complex numbers is real if and only if the corresponding points are collinear or concyclic; the inversion centered at the origin with radius 1 maps z to $\frac{1}{\overline{z}}$.

Denoting the complex coordinates of points as a, b, c and d we obtain:

$$(a^*, b^*; c^*, d^*) = \left(\frac{1}{\overline{a}}, \frac{1}{\overline{b}}; \frac{1}{\overline{c}}, \frac{1}{\overline{d}}\right) = \frac{\frac{1}{\overline{c}} - \frac{1}{\overline{a}}}{\frac{1}{\overline{c}} - \frac{1}{\overline{b}}} : \frac{\frac{1}{\overline{d}} - \frac{1}{\overline{a}}}{\frac{1}{\overline{d}} - \frac{1}{\overline{b}}} =$$
$$= \frac{\overline{c} - \overline{a}}{\overline{c} - \overline{b}} : \frac{\overline{d} - \overline{a}}{\overline{d} - \overline{b}} = \overline{\frac{c - a}{c - b} : \frac{d - a}{d - b}} = \frac{c - a}{c - b} : \frac{d - a}{d - b} = (a, b; c, d).$$

This allows to do the following useful remark:

12. Projection of a circle to itself. a) Let Ω be a circle, and S be an arbitrary point not lying on it. Map each point $X \in \Omega$ to the second common point $\mathcal{F}(X)$ of SX with Ω. Then the map $\mathcal{F}\colon \Omega \to \Omega$ is projective. **b)** Prove that the map transforming a point X to the line SX is NOT projective

13. Transfering of a circle to a tangent Let X move along a line ℓ touching a circle Ω. Let the second tangent from X to Ω touche it at $Y = \mathcal{F}(X)$. Then $\mathcal{F}\colon \ell \to \Omega$ is projective.

14. Transfering of a circle to itself via a line. Let X lie on a circle Ω, and ℓ be a fixed line. Let the tangent to Ω at X meet ℓ at point Z, and $Y = \mathcal{F}(X)$ be the base of the second tangent from Z toΩ. Then $\mathcal{F}\colon \Omega \to \Omega$ is projective.

The lemmas proved above help to solve problems using the projective movement.

15. Let γ_A, γ_B, γ_C — be the excircles of triangle ABC touching the sides BC, CA, AB respectively. Denote as ℓ_A the common external tangent of γ_B and γ_C distinct from BC. Define ℓ_B, ℓ_C similarly. Draw from a point lying on ℓ_A the tangent to γ_B distinct from ℓ_A and find its common point X with ℓ_C. Similarly find the common point Y of the tangent from P to γ_C and ℓ_B. Prove that the line XY touches γ_A.

16. An acute-angled triangle ABC with orthocenter H is inscribed into a circle ω centerd at O. A line l passes through H and meets the minor arcs AB and AC at points P and Q respectively. Let AA' be the diameter of ω. The lines $A'P$ and $A'Q$ meet BC at points K and L respectively. Prove that O, K, L and A' are concyclic.

17. Let S be the projection of the incenter I of an circumscribed quadrilateral $ABCD$ to the diagonal AC. Prove that $\angle BSA = \angle DSA$

18. A pentagon $ABCDE$ is circumscribed around a circle ω. The pairs of rays EA and CB, AE and CD, AB and DC, BC and ED meet at points P, Q, X, Y respectively. The circle ω touches AE at point R. It is known that $XY \parallel AE$. Let the circles $(AXQ), (PYE)$ meet at points S, T. Prove that S, T, R are collinear.

19. A line ℓ passes through the circumcenter O of triangle ABC and meets the sides AB, AC at points P, Q respectively. Prove that one common point of the circles with diameters BQ, CP lies on the nine-points-circle of triangle ABC, and the second one lies on its circumcircle.

20. 2023 lines concur. A circle is inscribed into each of 4046 formed angles in such a way that the circle inscribed into adjacent angles are tangent. A point is marked on each side of the angles. For each angle except one the segment joining the points marked on the sides of this angle touches the inscribed circle. Prove that this is correct for the remaining angle.

21. F ix the sidelines AB, AC and the incircle ω of triangle ABC. Prove that the touching point of ω with the side BC projectively depends on the Feuerbach point of this triangle

Theorem 2.2 *(May be used without proof) Let ABC be a triangle with circumcircle Ω and incircle (or excircle) ω. Let A' lie on Ω, and the tangents from it to ω meet Ω for the second time at B' and C'. Then the line $B'C'$ touches ω.*

Note that we have in this configuration a bijection: each point A of the circumcircle corresponds to the touching point of BC with the incircle. A hypothesis emerges that the corresponding map between to circles is projective.

Lemma 2.1 Homographies of Poncelet configuration. *Consider a Poncelet rotation of triangle ABC conserving the circumcircle and the incircle. Let A_1 be the touching point of side BC with the incircle, S_A and T_A be the midpoints of two arcs BC, and I_A be the center of the excircle touching the side BC, then all these points projectively depend on A.*

Call the map transforming A to A_1 as Poncelet homography and denote it as $\mathcal{P}$.

Mark the incenter I and the circumcenter O of triangle ABC. Note that

- The midpoint S_A of arc BC not containing A is the image of A in the projection of the circumcirle to itself centered at I.
- The point T_A is the image of S_A in the projection centered at O.
- Point A_1 is obtained from S_A by the homothety mapping the circumcircle to the incircle.
- By the trident theorem I_A is the image of the midpoint S_A of arc BC in the homothety centered at I with coefficient 2.

The homographies mapping A to S_A, T_A, A_1, I_A are presented, thus these points projectively depend on A

Usually the Poncelet homographies work effectively with the method of polynomial movement generalizing the projective movement. But it is possible to solve several difficult problems using the methods described above.

22. A convex hexagon $AQCPBR$ is inscribed into a circle Ω in such a way that the triangles ABC and PQR have a common incircle γ. A line ℓ parallel to BC and distinct from it touches γ. Let P_1 be the common point of ℓ and QR. Prove $\angle PAB = \angle P_1AC$.

23. Triangles ABC and DEF have a common incircle ω and circumcircle γ. Let L and K be the touching points of BC and EF respectively with ω, and M, N be the second common points of AL and DK respectively with γ, Prove that AM, EF, BC, ND concur.

24. Let the vertices A and B of a triangle projevtively move in the Poncelet rotation. Prove that this triangle is regular.

The last problem may seem too hard. We propose to return to it after the learning of *the relativity principle*.

It is known that the Poncelet theorem is correct not only for a triangle but for ane n-gon. So the following question is natural:

25. Consider a Poncelet rotation of polygon $A_1A_2...A_{2k+1}$. Does A_{k+1} projectively depend on the touching point of side A_1A_{2k+1} with the incircle?

Now we do not know an elementary proof of this assertion.

3 Conics

Definition 3.1 Conic. *An image of a circle in a projective map of the plane is called a non-degenerated conic. A degenerated conic is a pair of lines (possibly coinciding).*

You may use without proof that any non-degenerated conic is a circle, an ellipse, a parabola, or a hyperbola; and that any conic is the set of points (x, y) satisfying to an equation $ax^2 + by^2 + cxy + dx + ey + f = 0$, where a, b, c, d, e, f are real constants.

The following assertion may be used without proof:

Theorem 3.1 *There exists a unique conic passing through five point such that any three of them are not collinear.*

Definition 3.2 Cross-ratio of four points on the conic *Let A, B, C, D lie on a conic Ω, and S be an arbitrary point of this conic. The cross-ratio $(A, B; , C, D)$ on the conic equals to the cross-ratio $(SA, SB; SC, SD$.*

26. Prove that this definition is correct.

27. Three points A, B and C of a conic $\mathcal{C}$ are collinear. Prove that $\mathcal{C}$ is degenerated.

28. a) Lines a_t and b_t rotate projectively around points A and B respectively. Prove that the common point of a_t and b_t move projectively along some conic (probably degenerated) b) Points A_t and B_t move projectively along lines a and b respectively. Prove that the line A_tB_t envelops some conic (or passes through a fixed point).

When the conic in the previous problem is degenerated?

29. Points A_t and B_t move along two lines with fixed velocities. Which is the conic envelopped by the line joining these points?

Conics often appear as locus of points.

30. A circle and a line meeting at points A and B are give. Find the locus of points such that the tangent to the circle equals the distant to the line.

31. Similar isosceles triangles BA_1C, CB_1A, AC_1B have the sides of triangle ABC as the bases (All triangles are constructed inside ABC, or all triangles are constructed outside it). Prove that the lines AA_1, BB_1, CC_1 concur and find the locus of their common points.

The using of conics may be useful in problems not coherent directly wit them.

32. Let points P и Q be isogonally conjugated with respect to an acute-angled scalene triangle ABC. A point W is the midpoint of arc BAC of the circumcircle of ABC. The lines WP and WQ meet the circumcircle of ABC for the second time at points X and Y respectively. The lines passing through P and Q and parallel to AW meet AB, AC at points P_B, P_C, Q_B, Q_C. Prove that X, Y, P_B, P_C, Q_B, Q_C are concyclic.

33. The incircle of scalene triangle ABC touches BC, CA, and AB at points A_1, B_1, and C_1 respectively. Three flies creep along the lines AA_1, BB_1, and CC_1 with fixed velocities in such a way that in some moment they were at points A, B, and C, and in some other time they were at A_1, B_1, and C_1 respectively. In some time all three flies were collinear on a line p_1, and in some other time they were collinear on a line p_2. Prove that $p_1 \perp p_2$.

34. Points P, Q lie on the sides AD, CD of quadrilateral $ABCD$ in such a way that $\angle ABP = \angle CBQ$. Denote as S the common point of CP and AQ. Prove that $\angle PBS = \angle QBD$.

35. Let ABC be a triangle with incenter I. A line ℓ meets AI, BI, and CI respectively at points D, E, F distinct from A, B, C, and I. The perpendicular bisectors to segments AD, BE, and CF form a triangle Δ with incircle ω. Prove that ω and the circumcircle of ABC are tangent.

4 Polynomial dependences

Definition 4.1 Projective plane. *Projective plane is the set of all lines passing through a fixed point O of the space, these lines are called points of the projective plane.*

We can choose on any line in $\mathbb{R}^3$ passing through the origin an arbitrary point (x, y, z) distinct from the origin. So all non-zero triplets $[x : y : z]$ code points of the projective plane and are called homogenious coordinates. This definition naturally corresponds to the previous one: if a plane $\alpha|z = 1$ not containing O is fixed in the space, then each point A of this plane corresponds to the line OA, and each infinite point corresponds to the line passing through O, parallel to α, and having the same direction. The triple $[0 : 0 : 0]$ does not define any point of the projectibe plane.

Note that the axes OX and OY in the space are parallel to α, and the axis OZ is perpendicular this plane. Projecting OX and OY to α we obtain the standart coordinates in this plane. The point (x, y) of α has homogenious coordinates $[x : y : 1]$. The infinite points of the projective plane have homogenious coordinates $[x : y : 0]$.

Let p, q be two different points of the projective plane. Consider the plane passing through p, q, and O — it is defined by an equation $ax + by + cz = 0$, where a triple $[a : b : c]$ is defined up to a factor. We call $[a : b : c]$ the homogenious coordinates of a line. The triple $[0 : 0 : 0$ does not define any line.

It is clear that the point $[x_0 : y_0 : z_0$ lies on the line $[a : b : c$ if and only if $ax_0 + by_0 + cz_0 = 0$. From this we obtain by substitution the following

Lemma 4.1 *The coordinates of the line passing through the points $[x_1 : y_1 : z_1]$ and $[x_2 : y_2 : z_2]$ may be defined as*

$$[y_1z_2 - z_1y_2 : z_1x_2 - x_1z_2 : x_1y_2 - y_1x_2]$$

and the common point of the lines $[a_1 : b_1 : c_1]$ and $[a_2 : b_2 : c_2]$ has the coordinates

$$[b_1c_2 - c_1b_2 : c_1a_2 - a_1c_2 : a_1b_2 - b_1a_2]$$

Similarly we may define the homogenious coordinates on the projective line coding its points by the pairs $[x : y]$ defined up to factor. Since the time is a point of the projective line we can consider it as a pair $[t_1 : t_2]$.

Earlier we considered the maps from the projective line to the projective plane (and concretely to lines or conics on it) conserving the cross-ratios. Now we add the *polynomial maps* i.e. the functions $\mathcal{F}\colon \mathcal{RP}^1 \to \mathcal{RP}^2$ mapping a pair $[t_1 : t_2]$ to a triple of polynomials $P(t_1, t_2)$, $Q(t_1, t_2)$, $R(t_1, t_2)$ corresponding to he homogenious coordinates of a point of the projective plane.

Clearly the polynomials P, Q, R have to satisfy to several conditions. If this map cotrrectly transforms the points of projective line to the points of the projective plane the polynomials have to be homogenious ($t_1^2 + 2t_1t_2$ satisfies and $t_1^3 + 2t_1t_2$ does not satisfy) and they degrees have to be equal. Also all three polynomials may not equal zero, thus we suppose that they are relatively prime.

Definition 4.2 Power law *Say that the degree of dependence of point X on the time equals k, if the homogenious coordinates of X may be defined as $[P_1(t_1, t_2) : P_2(t_1, t_2) : P_3(t_1, t_2)]$, where P_i are relatively prime polynomials of degree k. The degrees of dependence of lines are defined similarly.*

It is easy to see that this definition is correct.

Lemma 4.2 Addition of degrees. *Let points X and Y move with degrees a and b respectively. Then the degree of the line XY is not greater than $a + b$.*

Denote the coordinates of X and Y as $[x_1 : x_2 : x_3]$ and $[y_1 : y_2 : y_3]$ respectively. Then the coordinates of XY are $y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1$. If the degrees of polynomials x_i are not greater than a, and the degrees of y_i are not greater than b, then the degrees of these polynomials are not greater than $a + b$. This estimation is not precise only if the obtained polynomials are not relatively prime.

Lemma 4.3 *Let a point X of degree 1 move one the line not passing trough a point S. Then the degree of line SX also equals 1.*

By the addition degree lemma the degree of SX is not greater than $0 + 1 = 1$. Since the line is not constant this estimation is precise.

In the following chapter we prove the general theorem: if the point X moves on an arbitrary trajectory, then the degree of SX equals to the degree of X.

Theorem 4.1 *The degree of point X projectively moving on a line equals 1. The degree of the line rotating projectively is also 1.*

Firstly demonstrate that any projective map between two lines may be presented as a composition of central projections and translations.

Let the map transform the points A_1, B_1, C_1 of line ℓ_1 to the points A_2, B_2, C_2 of ℓ_2. If $\ell_1 \parallel \ell_2$ project ℓ_1 to any line not parallel to it. Let a translation of ℓ_1 mapping A_1 to A_2 map B_1 and C_1 to B' and C' respectively. Denote as S the common point of $B'B_2$ and $C'C_2$ (may be infinite). The projection centered at S realize the required homography.

Clearly the translation does not change the degree of dependence. Hence we have to prove that that the projection does not change the degree. his follows from the previous lemma.

This theorem yields that if Y projectively depend on X and both points move on the lines then their degrees are equal. This is also correct for two rotating lines. Even more is true.

Theorem 4.2 *A homography of lines transforms the homogenious coordinates of their points linearly.*

This theorem may be used without proof.

Lemma 4.4 Redoubling of degree on a conic. *A point X moving projectively along a circle (a conic) has degree 2.*

Fix two points A, B on a circle, then the lines $a = AX$ and $b = BX$ rotate projectively, therefore X is the common point of two lines with degrees 1, and my the addition of degrees lemma we obtain the required assertion.

Similarly the homography of a line (a pencil) to a circle may at most redouble the degree of any point.

Remark: Clearly the dual lemma is also correct: let a line ℓ rotates around a circle (a conic) in such a way that the degree of touching point equals k. Then the degree of ℓ is not grater than $2k$

Theorem 4.3 *To prove the coincidence of two points with degrees k and l it is sufficient to verify $k+l+1$ dispositions.*

Not that the coincidence of points at time $[t_1 : t_2]$ is equivalent to the equality of the ratios of polynomials on t_1, t_2 defining the coordinates of points:

$$\begin{cases} \dfrac{P_x(t_1,t_2)}{P_y(t_1,t_2)} = \dfrac{Q_x(t_1,t_2)}{Q_y(t_1,t_2)} \\ \dfrac{P_x(t_1,t_2)}{P_z(t_1,t_2)} = \dfrac{Q_x(t_1,t_2)}{Q_z(t_1,t_2)} \end{cases}$$

This is equivalent to:

$$\begin{cases} P_x(t_1,t_2)Q_y(t_1,t_2) = P_y(t_1,t_2)Q_x(t_1,t_2) \\ P_x(t_1,t_2)Q_z(t_1,t_2) = P_z(t_1,t_2)Q_x(t_1,t_2) \end{cases}$$

And to prove the identity of two homogemious polynomials with degrees not grater than $k + l$ it is sufficient to verify $k + l + 1$ Not proportional dispositions. In fact if the polynomial equals zero at points (x_i, y_i) where $y_i \neq 0$, then dividing $P(t_1, t_2)$ onto t_2^d where d is the degree of polynomial, we obtain a polynomial on $\dfrac{t_1}{t_2}$ with the number of roots greater than its degree, hence it is equal to zero. If several y_i equal to zero, consider another time parameter $[\tau_1 : \tau_2]$ such that the infinite time τ does not correspond to any (x_i, y_i).

Similarly to prove that a line with degree k passes through a point with degree l it is sufficient to verify $k + l + 1$ dispositions.

The lemmas formulated above are «basic» and allow to solve many hard problems. Consider an example:

Example 4.1 *Points P and Q lie on the sideline BC of scalene triangle ABC in such a way that $BP = CQ$. Let ω be the incircle of the triangle, and ω_A be the excircle touching the side BC. Points S and T lie on ω and ω_A respectively in such a way that PS touches ω, and QT touches ω_A. Let AS and AT meet BC at points X and Y respectively. Probe that $BX = CY$.*

1. Choose a time and homogenious coordinates on the plane. Move P projectively along BC. Denote the degree of a point as d.
2. Point S projectively depend on P and moves along the circle, hence ny the redoubling of degree lemma $d(S) \leq 2$.
3. By the addition of degrees lemma $d(AS) \leq d(S) + d(A) \leq 2 + 0 = 2$
4. By the same lemma $d(X) \leq d(BC) + d(AS) \leq 0 + 2 = 2$. Similarly $d(Y) \leq 2$, because P and Q are symmetric with respect to the midpoint M of BC, i.e Q projectively depend on P.
5. Let $Y' = \mathcal{S}_M(X)$. Clearly $d(Y') = d(X) \leq 2$, thus we have to prove two points with degrees 2 coincide, for this it is sufficient to verify $2 + 2 + 1 = 5$ dispositions.
6. The dispositions $P = C, P = B, P = A_1$ — the touching point of the incircle, $P = M$, $P = \infty$ are clear.

Solving problems it is useful to understand how degenerated cases habe to be considered. Let in some moment we have to draw the line through two coinciding points: By the formula the coordinates of this line are $[0 : 0 : 0]$, which yields that all following polynomials equal zero. Hence the required assertion is always correct in degenereted dispositions. But we have to be accurate, consider the following example: a point X moves projectively on a conic, and S is a fixed point of this conic. Let we want to prove that several point Y of degree 1 lies always on SX. We have to interpretate correctly the line SX, when $S = X$. If we estimate the degree of SX by the degrees addition lemma its degree do not exceed 2, and if $X = S$ SX is the «zero line», this give us one disposition (because we did not estimate the degree precisely) and we have to find three other disposition. But if we consider SX as a line rotating projectively in the pencil of S, then we initially have to verify three dispositions but when $X = S$ the line has to be considered as the tangent and not the «zero line».

Now try to solve several exercises.

36. Points A and B move along two lines with constant velocities. Prove that the direction of line AB is projective.

37. Three points move projectively. How many positions have to be verified for prove that they are always collinear? The same question for three lines rotating projectively.

38. P oints X and Y move with degrees a and b respectively. Prove that the degree of the midpoint of segment is not greater than $a + b$.

39. Let P be a fixed point of a circle ω, and A move with degree a. Let B be such point that the arc PB is twice greater than the arc PA (we count the length of an arc counterclockwise). Prove that the degree of B is at most $2b$.

40. P olar transformations do not change the degree of dependence.

Now solve several problems.

41. Let $ABCDEF$ be a convex cyclic hexagon. The intersection of triangles ACE and BDF is a hexagon. Prove that its main diagonals concur.

42. Two perpendicular lines meet at the orthocenter of an acute-angled triangle. Prove that the midpoints of segments carving by these lines on the sidelines are collinear.

43. A triangle ABC and three collinear points P, Q, R are given. The lines AP, BP, and CP meet the circumcircle of ABC at points A', B', and C' respectively. The lines $A'Q, B'Q, C'Q$ meet the circumcircle at points A'', B'', and C''. The lines $A''R$, $B''R$, $C''R$ meet this circle at points A''', B''', and C'''. Prove that the lines AA''', BB''', CC''' concur at a point lying on the line passing through P, Q, and R.

44. A point X is marked on the circumcircle of triangle ABC. The lines BX and CX meet the altitudes CC_1 and BB_1 at points P and Q respectively. Prove that the midpoint of segment PQ lies on B_1C_1.

45. Let AH_A be the altitude of an acute-angled triangle ABC, and O be the center of its circumcircle Ω. The lines ℓ_A, ℓ_B, ℓ_C touch Ω at A, B, C respectively. Let S be the orthocenter of triangle formed by ℓ_A, ℓ_B, ℓ_C. Prove that the lines OH_A and SH_A are symmetric with respect to BC.

46. The excircle of triangle ABC centered at I_A touches the side BC at A_1 and touches the sidelines AB, AC at C_1, B_1 respectively. A point P of the line I_AC_1 is such that $AP \perp BI_A$. A point Q of the line I_AB_1 is such that $AQ \perp CI_A$. Prove that P, Q, A_1 are collinear.

47. Let X lie inside a scalene triangle ABC on its Euler line; and O be the circumcenter of ABC. The lines AX, BX, CX meet the opposite sides of ABC at A_1, B_1, C_1 respectively. Prove that the circle (AOA_1), (BOB_1), (COC_1) are coaxial.

48. A triangle ABC with orthocenter H is given. Points A_1, B_1, and C_1 kie on the circumcircle of the triangle in such a way that the lines AA_1, BB_1, and CC_1 concur. Denote as A_2, B_2, and C_2 the reflections of A_1, B_1, and C_1 about the midpoints of the corresponding sidelines. Prove that A_2, B_2, C_2, and H are concyclic.

49. A triangle ABC with orthocenter H is given. Points A_1, B_1, and C_1 kie on the circumcircle of the triangle in such a way that the lines AA_1, BB_1, and CC_1 concur. Denote as A_2, B_2, and C_2 the reflections of A_1, B_1, and C_1 about the corresponding sidelines. Prove that A_2, B_2, C_2, and H are concyclic.

50. The Turner theorem. Let points P, Q be inverse with respect to the circumcircle of triangle ABC, P_C be the reflection of P about AB, and P_CQ meet AB at point C'. Points A', B' are defined similarly. Prove that A', B', C' are collinear.

51. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D$. Prove that the midpoint of BD lies on a common internal tangent to the incircles of triangles ABC and ACD.

52. A triangle ABC is given. Denote as A_1 the common point of the medial line parallel to BC and the line joining the feet of altitudes to AB, AC. Points B_1, C_1 are defined similarly. Prove that the orthocenter of triangle $A_1B_1C_1$ lies on the Euler line of triangle ABC.

53. Let $ABCD$ be a cyclic quadrilateral, ω be its circumcircle, and P be the common point of its diagonals. Denote as I_A, I_B, I_C, and I_D the incenters of triangles APB, BPC, CPD, and DPA respectively. Let S_A, S_B, S_C, and S_D be the midpoints of «minor» arcs AB, BC, CA, and DA of ω. Prove that the lines I_AS_A, I_BS_B, I_CS_C, and I_DS_D concur.

5 Polynomial movement +

This chapter contains three subjects which be learned independently: generalizations of the theorems proved above, the relativity principle, and the combination of the movement with the Poncelet theorem.

To prove the advanced theorems concerning the polynomial movement we need to use the instruments most powerful than the real numbers. All definitions and theorems of the previous chapter are working for the complex numbers. Instead the real projective line $\mathbb{RP}^1$ we will consider $\mathbb{CP}^1$ — the set of lines in $\mathbb{C}^2$ passing through $(0, 0)$. Homogenious coordinates on the complex projective line are pairs $[z_1 : z_2]$ defined up to factor. The complex projective plane $\mathbb{CP}^2$, the homogenius coordinates on it, the pencils, etc. are defined similarly. Not that many results of previous chapters are correct for the complex numbers too. The unit comples circle is the set of solutions in $\mathbb{C}^2$ of the equation $x^2 + y^2 = 1$, or the set of points $[x : y : z]$ in $\mathbb{CP}^2$ such that $x^2 + y^2 = z^2$. Any non-degenerated conic may be obtained from a circle by a projective map (transforming the lines to the lines) and may be defined as the set of roots of an irreducible homogenious polynomial $P(x, y, z) = 0$.

54. Prove that a complex line and a non-degenerated complex conic have at most two common points. Do they intersect obviously?

The complex numbers are better than the real ones because any polynomial $f(x)$ distinct from a constant has a root (this may be used without proof). Furthermore any line of degree n on $\mathbb{CP}^2$ (i.e the set of roots of a homogenious polynomial of degree n) meets a line of degree m at mn points counting their multiplicity. This assertion is the general Bezou theorem. In this chapter we will use only the following patial case of the Besou theorem:

55. a) Prove that a homogenious non-constant polynomial $f(t_1, t_2)$ may be divided to some linear polynomial $at_1 + bt_2$ **b)**. Prove that any homogenious polynomial $f(t_1, t_2)$ may be uniquely (up to permutations and common factors) presented as a product $p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$, where p_i are homogenious linear polynomials.

Clearly the last problem is similar to the main theorem of the arithmetic.

Lemma 5.1 ***(The complex degrees addition lemma).*** *Let points X and Y move with degrees a and b respectively. Then the degree of the line XY is at most $a + b$, and if X and Y are distinct at any time, this estimation is precise.*

Denote the homogenious coordinates of points as $[x_1 : x_2 : x_3]$ and $[y_1 : y_2 : y_3]$ respectively, Then the line XY is defined by th coordinates $[y_1z_2 - z_1y_2 : z_1x_2 - x_1z_2 : x_1y_2 - y_1x_2]$. If the degree of x_i is at most a, and the degree of y_i is at most b the degrees of these polynomials do not exceed $a + b$. If this estimation is not precise then three obtained polynomials have a common divisor $d(t_1, t_2)$. Choose any linear divisor$p = at_1 + bt_2$ of d. Then at time $[b : -a]$ XY is «zero line»m i.e X and Y coinside, contradiction.

Lemma 5.2 *Let a point X of degree a move on a line not passing through a point S. Then the degree of line SX also equals a.*

Follows from the previous assertion.

Summing we obtain:

Theorem 5.1 *Let f be a homography of lines or pencils. Then f does not change the degree of dependence.*

Now we have to learn homographies using conics. We proved that the transfering of a point to a conic redoubles its degree. The inverse assertion is also true.

Theorem 5.2 ***(Throwing off conic).*** *Let a point X of degree k move on a conic, and S be fixed on it. Then the degree of line SX equals $\frac{k}{2}$, in partial k is even.*

You will prove this theorem solving the problems (we suppose that all object are complex).

56. A conic ω is given. Choose an arbitrary point S NOT on this conic and a line ℓ not passing through S. Let f be the projection of the conic from S to ℓ. Prove that any point of the line has two prototypes (excepting may be a finite set of points).

57. A point X moves polynomially on a conic ω. Choose an arbitrary point T ON ω and consider the projection of ω from S to the line $z = 0$. Let this projection map X to X'. So we defined the map g from the line of tome to the line $z = 0$. **a)** Prove that there exists a number k such that each point of line $z = 0$ (excepting may be a finite set) has exactly k prototypes in the map g. **b)** Prove that X coincide with almost any point of ω exactly k times.

58. Denote as X'' the image of X in the map f from the problem preceding the previous one. Count by two methods how many times X'' coincides with a general point of ℓ and obtain from this the theorem.

To start the next subject consider an example. Suppose that the vertices of any triangle move with degrees a, b, and c respectively. How can we estimate the degree of its circumcenter? It is not hard to see that the degree of the mid point of the segment joining two points with degrees a and b is at most $a + b$. The direction of the line passing through the points with degrees *aandbalsohasthedegreenotexceeding*a + b, *hencethedegreeoftheperpendicularbisectorisnotgreaterthan*(a + b) + (a + b) = 2(a + b)(*bytheadditiondegre* + c).*Meetingthesebisectorsandusingtheadditionprincipleweobtainthatthedegreeofthecircumcenterisnotgreat* + 2b + 2c.

This result sets thinking: the obtained estimation is not symmetric. Probably this means that the estimation is not precise. Furthemore we can suppose that the real estimation equals $2(a + b + c)$. The following instrument allows to prove this hypothesis.

Definition 5.1 Polynomial substitution *Suppose that subsets $P_1, P_2, ...P_n$ of projective plane are fixed, and for any points $p_1 \in P_1, p_2 \in P_2..., p_n \in P_n$ a point $[p_{0_x} : p_{0_y} : p_{0_z}] = \mathcal{R}(p_1, p_2, ..., p_n)$ is defined such that there exist homogenious polynomials R_x, R_y, R_z on $3n$ variables, $p_{0_x} = R_x(p_{1_x}, p_{1_y} ... p_{n_z})$, and p_{0_y}, p_{0_z} are defined similarly. Call such $\mathcal{R}$ a polynomial substitution on n points.*

Definition 5.2 Relative degree of dependence. *Let $\mathcal{R}$ be some polynomial substitution on two points, points q_1, q_2 move with several degrees, and $q = \mathcal{R}(q_1, q_2)$. Let r_1 be the minimal natural number such that for any time $[\tau_1 : \tau_2]$ the degree of $\mathcal{R}(q_1, q_2(\tau_1, \tau_2))$ is not greater than r_1 (the degree may be different for different $[\tau_1 : \tau_2]$). Call such r_1 the relative degree of dependence of q on q_1.*

The relative degrees of point depending on more than two points are defined similarly. Fix all q_i except one and see the degree of dependence of q on the remaining point.

59. Relativity principle. Let points $q_1, q_2 ..., q_n$ move polynomially, and the relative degrees of $q = \mathcal{R}(q_{1_t}, q_{2_t}, ...q_{n_t})$ equal $r_1, r_2, ...r_n$ for some polynomial substitution $\mathcal{R}$. Then the degree of q is not grater than $r_1 + r_2 + ... + r_n$.

60. Let the verices of a triangle move with degrees a, b, and c respectively. Then the degrees of the circumcenter and the orthocenter of this triangle are njt greater than $2(a + b + c)$.

61. Let points A and B projectively move along a conic ω. a) Prove that the line AB envelops some conic γ (or passes through a fixed point). b) Prove that ω and γ touch at two points of $\mathbb{CP}^{\not{2}}$ (i.e. the coordinates of touching points may be complex).

62. Let points A and B projectively move along a conic ω. Prove that the common point of tangents to the conic at A and B moves projectively along some conic or a line.

63. Let A move along a conic with degree $2a$, and B move along an arbitrary trajectory with degree b. Denote as C the second common point of line AB with the conic. Then the degree of C is not greater than $2a + 2b$.

64. Four points move with degrees a, b, c, and d. Then to prove that they are concyclic $2(a+b+c+d)+1$ disposition are sufficient.

It may be useful to apply the Poncelet theorem not only to the circumcircle and the incircle of a triangle, but also to its circumcircle and excircle. There are six degenerated disposition in this configuration: parametrizing a tangent ℓ to the excircle we can consider two dispositions when ℓ touches the circumcircle and the excircle, two disposition when it passes through the common points of these circles, and two disposition when the triangle is isosceles.

65. Denote the circumcircle and the excircle of triangle ABC as Ω and ω respectively. Let they meet at points X and Y. The common external tangents to Ω and ω touch Ω at points U and V. Prove that the tangents to ω at X and Y pass through U and V.

66. Let ω de the excircle of triangle ABC touching the side BC. The common external tangents to ω and (ABC) touch (ABC) at point X and Y. Prove that the line XY passes through the feet of bisectors from B and C.

67. The excircle ω of triangle ABC touches the side BC at point A_1 and meets the circumcircle of ABC at points X and Y. The tangents to ω at X and Y meet at point Z. Denote as S the midpoint of arc BAC. Prove that the lines SA_1 and AZ meet on the circumcircle of ABC.

68. The excircle ω of triangle ABC touches the side BC at point A_1 and meets the circumcircle of ABC at points X and Y. The lines A_1X and A_1Y meet the perpendicular bisector to the altitude from A at points U, V. Prove that AUA_1V is a parallelogram.

69. Let a line ℓ of degree d rotate around a point O. Fix two different vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with origin O. Prove that $\dfrac{\sin\angle(\ell, \overrightarrow{OA})}{\sin\angle(\ell, \overrightarrow{OB})} = P(t_1, t_2)$, where $P(t_1, t_2)$ is a homogenious polynomials with degree

d.

70. Let ω be the circumcircle of triangle, and Ω_A be the excircle touching the side BC. Denote the common points of these circles as X and Y. Let P and Q be the projections of A to the tangents at X and Y to Ω_A. The tangent to the circle (APX) at P meets the tangent to the circle (AQY) at Q at point R. Prove that $AR \perp BC$.