Inversions of the Feuerbach point, the poles of triangle and the Kulanin theorem

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Introduction

The Feuerbach point is one of well known triangle centers. It is defined as a tangency point of the incircle of triangle and its nine-points circle. The corresponding theorem was proved by Karl Wilgelm Feuerbach in 1822. After this more than 300 another proofs of this theorem were found, many of them use the inversion. The plurality of such proofs shows that the inversion is very useful instrument for analysis of properties of the Feuerbach point. In the first part of the project we propose new method of analysis of the Feuerbach point based on the considering of its reflections about several circles. This approach allows us to obtain several new nice results and to simplify the proofs of some known facts.

In the second part of the project we propose a method extending the obtaining results to more general configurations. We can say that several properties of the Feuerbach point are *partial cases* of general theorems, concerning absolutely distinct constructions which were not be used for its study. Starting from one nice assertion (the Kulanin theorem) we construct a theory generalizing different properties of the Feuerbach point.

The technic for solving the problems of the project is very manifold, thus we suppose that the following methods and constructions are known to the participants:

- inversion and its properties;
- poles and polars, harmonic quadruples;
- linear moving of points;
- properties of conics (not necessary);
- complex numbers (not necessary).

Fix the notations which will be used in all problems of the project.

- A, B, C — the vertices of triangle;
- M_a, M_b, M_c — the midpoints of segments BC, AC, AB respectively;
- H_a, H_b, H_c — the feet of altitudes of triangle ABC from A, B, C respectively;
- L_a, L_b, L_c — the feet of bisectors of triangle ABC from A, B, C respectively;
- ω — the incircle of triangle ABC centered at I;
- G_a, G_b, G_c — the tangency points of the incircle of triangle ABC with sides BC, AC, AB respectively;
- G'_a, G'_b, G'_c — the tangency points of the excircles of triangle ABC with sides BC, AC, AB respectively;

- K_a, K_b, K_c — the reflections of G_a, G_b, G_c about lines AI, BI, CI respectively;
- λ_a — the circle with diameter BC;
- S_{ab}, S_{ac} — the Sharygin points, i.e. the common points of λ_a with medial lines M_aM_b and M_aM_c respectively;
- ε, ε_a — the nine-points circles of triangles ABC and IBC respectively;
- F — the tangency point of the incircle and the nine-points circle of triangle ABC (the Feuerbach point).

0. Auxiliary facts

The problems of this section contain known facts and it is not necessary to present their solutions. But we recommend to solve the problems unknown to you. This will help to solve the remaining problems of the project.

0.1 (The nine-points circle) Prove that the midpoints of sides of triangle, the feet of its altitudes, and tyhe midpoints of segments joining its vertices with the orthocenter are concyclic.

0.2 (The Euler line) Prove that the center E of the nine-points circle, the circumcenter O, the orthocenter H, and the centroid M are collinear, and $HE : EM : MO = 3 : 1 : 2$.

0.3 (The Mickel point) Consider four lines in general position forming four triangles. Prove that the circumcircles of these triangles have a common point.

0.4 Prove that the Euler line of triangle $G_aG_bG_c$ passes through O.

0.5 (The Simson line) Prove that the projections of a point lying on the circumcircle Ω of triangle ABC to its sidelines are collinear.

0.6 Prove that the Simson line of point P bisects segment PH.

0.7 (Isogonal conjugation) Consider a triangle ABC and a point P distinct from its vertices. Prove that the reflections of lines AP, BP, and CP about the bisectors AI, BI, CI respectively concur at a point P'.

0.8 Prove the following properties of isogonal conjugation:

- the points O and H are isogonally conjugated;
- the pedal circles (i.e. the circles passing passing through the projections of the given points to the sidelines of triangle) of isogonally conjugated points P and P' coincide, and this circle is centered at the midpoint of segment PP';
- the point isiogonally conjugated to a point P of the circumcircle is the infinite point of the perpendicular to the Simson line of P;
- the point A is isogonally conjugate to points of line BC distinct from B and C.

0.9 Let P and Q be isogonally conjugated, and P_b, P_c, Q_b, Q_c be their projections to AC and AB respectively. Prove that P_bQ_c, P_cQ_b, and PQ concur.

0.10 Prove that any conic passing through A, B, C, and H is an equilateral hyperbola.

0.11 Prove that the isogonal image of a line is a conic.

1. The Feuerbach theorem

1.1 Prove that the triplets of lines (G_bG_c, BI, M_aM_b) and (G_bG_c, CI, M_aM_c) concur at points S_{ab} and S_{ac} repectively. These points lie on the circle λ_a with diameter BC.

1.2 Prove that the angle between the line BC and the nine-points circle ε equals to $|\angle B - \angle C|$.

1.3 Prove that the quadruple (H_a, L_a, G_a, G'_a) is harmonic.

1.4 Prove the Feuerbach theorem, i.e. the tangency of the incircle ω and the nine-points circle ε

1.5 Generalize the proof from the previous problem to the case of the excircles. Prove that the triangle formed by the tangency points of the nine-points circle with the excircles is perspective to the triangle $L_aL_bL_c$ with center at F.

1.6 (*) Find another remarkable circles tangent to the nine-points circle. Which interesting properties of tangency points can be noted?

1.7 Prove that the line FG_a bisects angle H_aFM_a.

1.8 Prove that the nine-points circles of four triangles formed by arbitrary four points in general position have a common point (the Poncelet point).

1.9 Let P be an arbitrary point distinct from O and H. Prove that the pedal circle of P passes through the Poncelet point of $ABCP$.

1.10 Prove that the triangles $K_aK_bK_c$ and $M_aM_bM_c$ are homothetic with center F.

1.11 Prove that F lies on the nine-points circle ε_a of triangle BIC.

1.12 Let $\widetilde{G}_a$ be the point of ω opposite to G_a. Prove that the line $F\widetilde{G}_a$ bisects segment AI.

2. Inversion images of the Feuerbach point

Now pass to the substantial part using the inversion. Consider again the circle λ_a with diameter BC. Let F'_a be the reflection of F about λ_a. The point F'_a will be the main object of studying in this section.

Also we will actively use in the problems of this part the points K_a, K_b, and K_c symmetric to G_a, G_b, and G_c with respect to the bisectors of the corresponding angles.

2.1 Prove that F'_a is the radical center of circles λ_a, ε, and ε_a.

2.2 Let K'_b and K'_c be the common points of rays K_aK_b and K_aK_c respectively with the line H_bH_c. Prove that the fives of points $(F, F'_a, K_b, K'_b, S_{ab})$ and $(F, F'_a, K_c, K'_c, S_{ac})$ are concyclic on circles ψ_{ab} and ψ_{ac} respectively.

2.3 Prove that the line M_aS_{ab} touches the circle ψ_{ab}.

2.4 Prove that the line $S_{ab}S_{ac}$ bisects the angle $K_bF'_aH_b$.

2.5 Prove that F'_a is the common point of lines K_bK_c, L_bL_c, and $G'_bG'_c$.

2.6 Let T_a be the common point of lines FG_a and $S_{ab}S_{ac}$. Then the quadruples (F, T_a, M_b, S_{ab}) and (F, T_a, M_c, S_{ac}) are concyclic on circles ψ'_{ab} and ψ'_{ac} respectively.

2.7 Prove that F is the Mickel point for the triangle $M_aM_bM_c$ and the line $S_{ab}S_{ac}$.

2.8 Prove that T_a, T_b, and T_c lie on medial lines M_bM_c, M_cM_a, and M_aM_b respectively.

2.9 Consider an arbitrary triangle $\triangle = P_aP_bP_c$ homothetic to the triangle $M_aM_bM_c$ with center at F. Consider circles $\psi^{\triangle}_{ab}$ and $\psi^{\triangle}_{ac}$ passing through (F, P_b, S_{ab}) and (F, P_c, S_{ac}) respectively. Prove that their common point distinct from F coincides with the common point $A^{\triangle}$ of lines P_bP_c and $S_{ab}S_{ac}$.

2.10 Prove that when $A^{\triangle}$ coincides with G_b the circle $\psi^{\triangle}_{ac}$ passes through F, H_b, G_b, S_{ac}, S_{ca}, and the reflection of I about $S_{ac}S_{ca}$.

2.11 (*) Prove that the perpendiculars to lines BC, CA, and AB from their common points with $FA^{\triangle}$, $FB^{\triangle}$, $FC^{\triangle}$ respectively concur at point lying on OI. Also the circle passing through these common points passes through F.

For which triangle $\triangle$ the configuration of problem **2.9** passes to the configurations of problems **2.2** and **2.6** ?

The last problem is very difficult. The following section is dedicated to the construction allowing not only to solve this problem but to generalize it to similar configurations.

3. General poles of triangle and the Kulanin theorem

In this section we generalize the notion of Feuerbach point and prove the assertions presented above (partially in problem **2.11**) for general cases. We will use next additional notations.

- O — the center of circumcircle Ω of triangle ABC;
- ℓ — an arbitrary line passing through O;
- A_ℓ, B_ℓ, C_ℓ — the common points of ℓ with BC, CA, and AB respectively;
- $P_aP_bP_c$ — the pedal triangle of P with respect to triangle ABC, and Ω_P — its pedal circle;
- F_ℓ — general Feuerbach point (see. the Kulanin theorem).

To formulate the necessary notions we use the following theorem. Now it can be used without proof.

Theorem 1 (Kulanin)**.** Let P be an arbitrary point of line ℓ passing through O. Then all pedal circles Ω_P have a common point.

3.1 (*) Prove this theorem.

Denote by F_ℓ the common point of pedal circles Ω_P of points lying on ℓ. We will call F_ℓ the *general Feuerbach point*. This point depends on line ℓ passing through the circumcenter O of triangle ABC. In the case $\ell = OI$ the point F_ℓ coincides with the Feuerbach point F. Note also that all points F_ℓ lie on the nine-points circle ε of triangle ABC (because the nine-points circle is the pedal circle of O).

3.2 Consider a triangle ABC, a line ℓ passing through O, and a point P on it. Denote by A_{pp} the common point of lines P_bP_c and M_bM_c. Prove that P_a, A_{pp}, and F_ℓ are collinear.

3.3 (*) (Main theorem) Consider an arbitrary line ℓ passing through the circumcenter O of triangle ABC. Let P and Q be arbitrary points of this line. Note as A_{pq} the common point of lines P_bP_c and $F_\ell Q_a$. Define points B_{pq} and C_{pq} similarly. Take an arbitrary point R on ℓ and draw the lines through A_{pq}, B_{pq}, C_{pq} parallel to R_bR_c, R_cR_a, R_aR_b respectively. Then the obtained triangle is homothetic to triangle $R_aR_bR_c$ with homothety center F_ℓ.

3.4 Using the main theorem prove the results of problems **2.5** , **2.8** , **2.11** .

3.5 Prove that the circumcircle of triangle formed by the lines $A_{pq}A_{qp}$, $B_{pq}B_{qp}$, $C_{pq}C_{qp}$ passes through F_ℓ.

3.6 Prove that the reflections of ℓ about the medial lines of triangle ABC concur at point F_ℓ.

3.7 Let A_ℓ, B_ℓ, and C_ℓ be the common points of ℓ with BC, CA, and AB respectively. Prove that the circles with diameters AA_ℓ, BB_ℓ, CC_ℓ have two common points F_ℓ and F'_ℓ, F'_ℓ lies on the circumcircle Ω of triangle ABC, and the orthocenter H of this triangle lies on the line $F_\ell F'_\ell$.

3.8 Let OH be the Euler line of triangle ABC. Prove that the triangle formed by the poles A_{hh}, B_{hh}, C_{hh} is autopolar with respect to the nine-points circle ε, in such a way that the vertices of ABC lie on the sidelines of $A_{hh}B_{hh}C_{hh}$.

3.9 Let triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ be perspective to ABC and their vertices lie on the corresponding sidelines of ABC. Let A_{pq}, B_{pq}, C_{pq} be the common points of the corresponding sidelines of triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$. Prove that:

a) The vertices of ABC lie on the sidelines of triangle $A_{pq}B_{pq}C_{pq}$;

b) Triangle $A_{pq}B_{pq}C_{pq}$ is autopolar ith respect to conic Ω passing through P_a, P_b, P_c, Q_a, Q_b, Q_c;

c) Triangle $A_{pq}B_{pq}C_{pq}$ is perspective to triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ and the corresponding perspective centers lie on the conic Ω.

3.10 Let X_a be an arbitrary point of line BC. Denote by X_b the common point of lines X_aB_{hh} and AC, note as X_c the common point of lines X_aC_{hh} and AB. Prove that

- the line X_bX_c passes through A_{hh};
- the lines AX_a, BX_b, and CX_c concur at point X;
- the circumcircle $X_aX_bX_c$ passes through F_{OL}, where L is the Lemoine point of ABC.

Finally formulate several open problems.

3.11 The definition of poles A_{pq}, B_{pq}, C_{pq} depends on the order of choosing of points P and Q lying on ℓ. The question: how the poles A_{pq} and A_{qp}, i.e. the points of the same points taken in different order are correlated? We have the following hypothesis. *The common points Z_a, Z_b, Z_c of the corresponding sidelines of triangles $A_{pq}B_{pq}C_{pq}$ and $A_{qp}B_{qp}C_{qp}$ form a triangle homothetic to ABC with center F_ℓ.*

3.12 It is interesting to find such pair of points P and Q of line ℓ that the vertices of ABC lie on the sidelines of triangle $A_{pq}B_{pq}C_{pq}$. Is always possible to choose such pair of points of ℓ? Experiments show that such points always exist, but they can not lie too far from O. How can we describe such positions of P and Q that the vertices of ABC lie on the sidelines of triangle $A_{pq}B_{pq}C_{pq}$?