Discharging method*

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Introduction

This project is devoted to a certain classical idea in graph theory. It plays a crucial role in the proofs of many "structural" results on graphs drawn in the plane that are not necessarily planar. This idea became well-known because of its application in the proof of the celebrated four color theorem. This idea is the so-called *discharging method*: it is a certain form of *double counting* based on Euler's Formula for a planar graph.

Our project is aimed at an experienced problem solver. We do not plan to show step-by-step instructions and explain how considered method works. Instead, we are going to provide sufficient amount of problems of gradually increasing difficulty, which will form a ladder both to understanding the method and to the valued results achieved throw it. Below you can find three interesting problems of the project, which are very difficult, and so we recommend you to postpone solving them for now and to return to them only after the semifinal.

1. Let P be a finite set of points in the plane. P is called a *magical configuration* if there is an assignment of positive weights to the points of P such that, for every line ℓ containing at least two points of P, the sum of the weights of all points of P on ℓ equals 1. Describe all magical configurations.

2. We say that a graph Γ is *drawn in the plane* if the vertices $V(\Gamma)$ are represented by distinct points and the edges $E(\Gamma)$ are represented by (Jordan) arcs, each connecting two vertices and containing no other vertex. We say a graph is *quasi-planar* if it can be drawn in the plane in such a way that no 3 of its edges are pairwise crossing in their inner points[1]. Prove that, for a quasi-planar graph Γ, the inequality $|E(\Gamma)| \leq 8|V(\Gamma)| - 20$ holds.

3. A graph Γ is called a *matchstick graph* if Γ can be drawn in the plane in such a way that edges are represented by segments of length 1, and no two edges cross. Prove that a matchstick graph must have a vertex of degree different from 5.

Also, it is worth mentioning that in the last section of our project (that will be given to you after the semifinal) you can find some open research problems that probably could be solved using this method. A solution of any of them is a publicable result.

*Here you can find the current results: https://clck.ru/HMnm8

[1]Note that if one replaces 3 with 2 in the above definition, one gets the definition of a planar graph. This explains the choice of the term "quasi-planar".

Useful notations

Denote by $V(\Gamma)$ and $E(\Gamma)$, respectively, the sets of the vertices and the edges of a graph[2] Γ. Moreover, if an embedding of Γ in the plane is given, then $F(\Gamma)$ denotes the set of the faces of the drawing of Γ. Famous *Euler's Formula* states that, for a connected planar graph Γ,

$$|V(\Gamma)| - |E(\Gamma)| + |F(\Gamma)| = 2.$$

Denote the *degree* of the vertex $v \in V(\Gamma)$ by $\deg(v)$. Also, denote by $\delta(\Gamma)$ and $\Delta(\Gamma)$ the *minimum* and *maximum* degree of a vertex in $V(\Gamma)$, respectively. A graph is called *n-regular* if the degree of any vertex is n. *The weight* of an edge uv is $\deg(u) + \deg(v)$. Call *the degree* $\deg(f)$ of a face f the number of edges along the boundary of f (that is a cut-edge is counted twice).

Definition. We say that a graph has a *proper coloring* in k colors (or it is *k-colorable*) if for each vertex one can assign one of the numbers $1, \ldots, k$ (called *colors*) in such a way that adjacent vertices are colored differently. The chromatic *number* $\chi(\Gamma)$ is the least k such that a graph G has a proper coloring in k colors.

A subgraph Γ' of a graph Γ is *induced* if $E(\Gamma')$ contains all edges of $E(\Gamma)$, that connect vertices of $V(\Gamma')$.

1 Double counting

1.1. Some cells of a given table are marked. For every marked cell, the number of marked cells in its column equals to the number of the marked cells in its row. Prove that the number of rows with at least one marked cell is equal to the number of columns with at least one marked cell.

1.2. E elves and D dwarfs arrived to the Zilantkon convention. It turns out that during the convention every dwarf got into a fight with at least one elf and every elf got into a fight with at most ten dwarfs. Also, it is known that every dwarf has more opponents-elves than any of his opponents-elves has opponents-dwarfs. Prove that $D \leq \frac{10}{11}E$.

1.3. A table has m rows and n columns, where $m < n$. Some cells are marked in such a way that every column contains at least one marked cell. Prove that there is a marked cell such that the number of marked cells in its row is larger than the number of marked cells in its column.

1.4. In a school library there were exactly k empty bookshelves yesterday. Today in the morning, some books were rearranged in such a way that every shelf is not empty anymore. A book is called *boring* if the number of books on its current shelf is less than the number of books on its shelf before the rearrangement. Prove that there are at least $k + 1$ boring books.

1.5. Suppose that an $n \times n$ table is filled **a)** with numbers 0 and 1 **b)** with non-negative integers in such a way that if some cell of the table contains 0, then the sum of all numbers in its cross[3] is at least 1000. Find the least possible sum of numbers in the table.

[2] From now on, we consider graphs without loops and multiple edges unless otherwise stated.

[3] A cross of a cell is the union of cells lying with it in the same row or the same column

1.6. Suppose that a convex n-gon and m red points distinct from the vertices of the polygon are drawn on a blackboard. It turns out that each segment between two vertices of the polygon contains at least one red point. Prove the inequality

$$m \geq n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\lfloor (n-1)/2 \rfloor}\right).$$

1.7. Consider n unit circles drawn in the plane. It is known that each of them intersects with at least one another circle and also there are no two touching circles. It is possible that more than two circles pass through one point. Prove that there are at least n intersection points.

1.8. A square is cut into several triangles. Prove that there are two triangles sharing a common edge.

1.9*. There are n lines in general position in the plane, that is, no three of them share a common point and no two are parallel. These lines split the plane into several parts. Prove that there are at least $n-2$ triangles among them.

2 Problems about graphs

Planar graphs

This subsection contains problems that historically were proven by the discharging method. Most of them are lemmas in serious articles. However, these problems are not harder than difficult olympiad ones. Thus, we find it reasonable to present them without extra hints.

2.1. Let V be a convex polyhedron without 4-gonal and 5-gonal faces. Prove that V has at least 4 triangular faces.

2.2. Let Γ be a planar graph with $\delta(\Gamma) \geq 2$ and all cycles of length at least 7. Prove that there is an edge of Γ of weight at most 5.

2.3. Let Γ be a planar graph with $\delta(\Gamma) \geq 5$ such that all its faces are triangles, and there is no two adjacent vertices of degree 5. Prove that there is a face with degrees of vertices 5, 6 and 6, respectively.

2.4. Let Γ be a planar graph with $\delta(\Gamma) \geq 5$. Prove that there is an edge of weight at most 11.

2.5. Let Γ be a planar graph such that $\delta(\Gamma) \geq 3$. Prove that there are a face f and a vertex $v \in f$ such that either $\deg(v) = 3$ and $\deg(f) \leq 5$ or $\deg(v) \leq 5$ and $\deg(f) = 3$.

2.6. Let Γ be a connected planar graph such that $\delta(\Gamma) \geq 3$. Prove that there is a face of length at most 5 with the degrees of all its vertices but one do not exceed 11.

Light colorings

Every problem of the previous subsection states that a graph with some global property satisfies some local property. For instance, an information about the minimal degree of a graph implies that it contains a given subgraph. Analogous statements can be useful for proving of upper bounds on the chromatic number of a graph.

2.7. **a)**Prove that any planar graph is 6-colorable.

b) Suppose that, for any subgraph of a given graph Γ, one of the following conditions holds: 1) there is a vertex of degree at most $d-1$; 2) there is an even induced cycle such that the degree of every its vertex does not exceed d. Prove that the induced graph Γ is d-colorable.

Definition. We say that a graph has a *very light coloring* in d colors if one can consequently delete all the vertices in such a way that, at each step one deletes a vertex of degree at most $d-1$.

Note. For instance, Problem 2.7a is about a very light coloring.

Definition. A graph is called *k-choosable* if for any assignment of sets (called *lists*) of k colors to vertices (the lists are not necessary the same), there exists a proper coloring according to the lists, that is, for each vertex one can choose a color from its list in such a way that the colors of adjacent vertices are different.

Analogously we can define *k-edge-choosability.*

The *list chromatic number* of Γ, written (Γ), is the least k such that Γ is k-choosable.

It is obvious that k-choosability of a graph implies its k-colorability. Indeed, it is possible that all the lists assigned to the vertices are the same. It turns out that the converse is false, that is, a graph can be k-colorable but not k-choosable.

2.8. **a)**Prove that for a given k there exists an integer n such that the complete bipartite graph $K_{n,n}$ is not k-choosable. Although it is 2-colorable as any bipartite graph.

b) Prove that any even cycle is not only 2-colorable but also 2-choosable.

2.9. **a)**Suppose that every inner face of a planar graph Γ is a triangle and its outer face is bounded by a cycle $v_1 \dots v_k v_1$. Suppose further that with v_1 and v_2 lists of 2 colors are associated, with every other vertex of the outer face a list of 3 colours, and with every inner vertex a list of 5 colours. Then there exists a proper list coloring of Γ for the given lists.

b) Every planar graph is 5-choosable.

Definition. We say that a graph is *lightly d-choosable* with respect to its induced subgraphs $\Gamma_1, \dots, \Gamma_k$ if one can consequently delete these induced subgraphs to get the empty graph in such a way that, at i-th step the number of edges of every vertex of Γ_i connecting it with the remaining vertices but $V(\Gamma_i)$ is at most $d - (\Gamma_i)$.

Notice that the graph in Problem 2.8b is lightly d-choosable.

It is possible to use the idea generalizing light choosability for proving the existence of total coloring.

Definition. Given an assignment of lists of colors to vertices and edges of a given graph Γ (the sizes of lists are not necessary the same), we say that Γ has a *proper total list coloring* if for every vertex and every edge, one can choose a color from its list in such a way that any two neighboring elements[4] receive different colors.

[4]Two vertices are called *neighboring* if they are connected by an edge, two edges are *neighboring* if they share a common vertex, a vertex and an edge are *neighboring* if the vertex belongs to the edge.

Consider a planar graph Γ with $\Delta(\Gamma) \geq 11$. Assign lists of size $\Delta(\Gamma)$ to the vertices of Γ and lists of size $\Delta(\Gamma) + 2$ to the edges. Prove that Γ has a proper total list coloring for this set of lists.

3 Main problems

In this section, we present four problems proved by the discharging method that were results of mathematical articles. We reveal a few intermediate steps of the solution.

3.1 Matchstick graphs

Definition. A *matchstick graph* is such a graph whose vertices are points in the plane, edges are represented by segments of length 1, and no two edges cross. (Note that a pair of vertices at distance 1 is not necessary an edge.)

Key Problem 1. Prove that a 5-regular matchstick graph does not exist.

Note. Actually, in any matchstick graph there is a vertex with degree less than 5.

Definition. A graph Γ is called *a graph of minimal distances*, if Γ can be drawn in the plane in such a way that edges are pairs of points at distance 1, and no pair of vertices are at distance less than 1.

3.1.1. Let Γ be a graph of minimal distances on n vertices that are in general position[5]. Prove that **a)** $E(\Gamma) < 5n/2$, **b)** there exists a constant $c < 5/2$ such that $E(\Gamma) \leq cn$.

c) An *interesting graph* is a graph of minimal distances such that, for every vertex, it and its neighbors are in general position[6]. Prove that, for every $c < 5/2$, there is an interesting graph Γ with at least $c|V(\Gamma)|$ edges.

3.1.2. Solve Key Problem 1.

3.1.3*. Prove that a 4-regular matchstick graph contains at least 20 vertices.

3.2 Quasi-planar graphs

Definition. We say that a graph Γ is *drawn in the plane* if the vertices $V(\Gamma)$ are represented by distinct points and the edges $E(\Gamma)$ are represented by (Jordan) arcs, each connecting two vertices and containing no other vertex.

This definition is auxiliary, it is needed to define different classes of graphs. For instance, prohibiting edge crossings, we obtain the definition of planar graphs. Considering different relaxations of the crossing condition[7] we obtain different extensions of the class of planar graphs.

[5] that is, no three of them are on the same line

[6] all vertices are not necessary in general position

[7] For example, the condition that each edge has at most t inner crossings, or that there are at most d edges pairwise crossing in inner points

Definition. A graph is called *quasi-planar* if it can be drawn in the plane in such a way that each two edges intersect in at most one inner point and no three edges are pairwise crossing in their inner points.

Key Problem 2. a) Prove that $|E(\Gamma)| \leq 8n - 20$ for a quasi-planar graph Γ on n vertices.

b) Try to improve the statement of the previous problem. Both sharpening of the inequality and its generalizations for other classes of graphs are interesting.

3.2.1. Suppose that the graph satisfies the following additional condition: There are no three edges e_1, e_2, e_3 such that e_1 and e_2 has the same end A, the edge e_3 intersects them at interior points B and C, and there are no other intersections on the sections of the arcs AB and AC. Prove that the number of edges is at most $4n - 8$.

3.2.2. (Исправить бы формулировку) Without the additional condition in the last problem, prove the bound for the number of edges $10n - 20$. In the solution of the previous problem you have found the initial distribution of charges that very likely also works here.

3.2.3. Finally, solve Key Problem 2a. In the solution of the previous problem you have found the initial distribution of charges that very likely also works here.

3.3 List edge coloring

Here, we use the concept of light coloring in a new setting. Recall that the idea is to show that it is possible to delete a part of a given graph, and thus to reduce the question of coloring of the graph to the same question for the remaining subgraph.

In this section, by *triangle* we denote a 3-cycle. Denote by Γ a planar graph with an extra condition: There are no two triangles sharing an edge.

The main goal of this section is to solve the following problem.

Key Problem 3. Prove that if $\Delta(\Gamma) \geq 6$, then Γ is $(\Delta(\Gamma) + 1)$-edge-choosable.

Note. The statement of Key Problem 3 holds in the case $3 \leq \Delta(\Gamma) \leq 4$, even for non-planar graphs. If $\Delta(\Gamma) = 5$, then Γ is $(\Delta(\Gamma) + 2)$-edge-choosable.

3.3.1. Prove that if $\Delta(\Gamma) \geq 7$, then Γ is $(\Delta(\Gamma) + 1)$-edge-choosable.

You solved Key Problem 3.3.1 in the case $\Delta(\Gamma) \geq 7$. To solve the case $\Delta(\Gamma) = 6$, additional efforts are required.

3.3.2. There is an extra configuration that is special for the case $\Delta(\Gamma) = 6$: A 6-vertex that is incident to three triangles, two of these triangles are of type $(6, 6, 3)$ and the third one is of type $(6, 6, 3)$, $(6, 5, 4)$, or $(6, 6, 4)$. Prove that if Γ contains such a configuration, then it is possible to reduce the question on coloring of Γ to the same question for its subgraph.

3.3.3. Finally, prove that if $\Delta(\Gamma) = 6$, then Γ contains one of some three configurations[8] each of those allows to reduce the question on coloring of Γ to the same question for its subgraph.

[8] Solving problem 3.3.1, you found the first configuration, another one is defined in problem 3.3.2. Try to figure out what is the third one.

3.4 Magical configurations

Definition. A finite planar point set P is called a *magical configuration* if there is an assignment of positive weights to the points of P such that, for every line ℓ containing at least two points of P, the sum of the weights of all points of P on ℓ equals 1.

Key Problem 4. Describe all magical configurations.

This problem is partially motivated by famous Silvester's Theorem.

Sylvester's Theorem. *If P is a finite set of points in the plane, then either P is a subset of a line, or there exist two points $A, B \in P$, such that the line AB contains no other points of P.*

We need the following interesting configuration.

Definition. A *Failed Fano* configuration is the following configuration of 7 points in the plane: Points A_1, A_2, A_3, A_4 that are in general position, and the other three are intersections of lines: $B_1 = A_1A_2 \cap A_3A_4$, $B_2 = A_1A_3 \cap A_2A_4$, and $B_3 = A_1A_4 \cap A_2A_3$.

It turns out that the dual language is more convenient to solve Key Problem 4.

3.4.0. **a)** *Optional problem for those who use the language of projective geometry.* Consider a duel configuration to a magical configuration of points in the projective plane (that is a configuration of lines). Describe a configuration dual to a Failed Fano configuration.

b) *Optional problem for those who do not use the language of projective geometry.* Suppose that points and lines are drawn in the plane. Find a correspondence π that associate a drawing in the plane with a drawing on a unit sphere such that the image of a point is an equator[9] of the sphere, the image of a line in the plane is a pair of antipodal points on the sphere (note that the image of a line is not the union of the images of its points!), and the correspondence preserves the incidence relation, that is, if a point A lies on a line ℓ, then $\pi(\ell)$ belongs to the equator $\pi(A)$. Prove other natural properties of the correspondence:

- The image of a line AB is a pair of the intersections of equators $\pi(A)$ and $\pi(B)$.
- The image of the intersection of lines ℓ_1 and ℓ_2 is an equator containing pairs of antipodal points $\pi(\ell_1)$ and $\pi(\ell_2)$.

Also, find the image of a Failed Fano configuration.

3.4.1. Using dual terms and Euler's Formula, prove Silvester's Theorem.

A configuration of lines (equators) is *magical* if it is dual to a magical configuration of points. From now on, we consider only magical configurations of lines.

3.4.2. Suppose that, in a magical configuration P, the assigned weight to a line (equator) is greater than $\frac{1}{2}$. Prove that all the rest lines (equators) of P share a common point (pair of antipodal points).

From now on, we suppose that there is no point A belonging to all lines but, maybe, one. A *red line* is a line with assigned number $\frac{1}{2}$. All the rest lines are called *blue*.

[9] An *equator* is the intersection of a sphere with a plane passing through its center.

3.4.3. Prove that there exists a point A that belongs to exactly two blue and one red lines (equators). Recall that the configuration consists of lines (equators).

3.4.4. Prove that there is a blue quadrilateral with red diagonals.

From now on, we call such a quadrilateral *bad.*

3.4.5. Prove that there is a blue triangle sharing a common edge with a bad quadrilateral.

From now on, we call such a triangle *bad.*

3.4.6. Prove that if a bad triangle shares two edges with bad quadrilaterals, then the configuration is the configuration dual to a failed Fano configuration.

From now on, we suppose that each bad triangle shares an edge with exactly one bad quadrilateral.

3.4.7. Prove that there are a bad triangle t, its bad quadrilateral d, and one of their common vertices denoted A such that the cell, vertical[10] to t with respect to A, is a quadrilateral.

3.4.8. Consider the partition of the projective plane (the sphere) by the blue lines (equators). Prove that there is a blue line (a blue equator) bounding exactly two (four) triangles.

3.4.9. Prove that the case described in the previous problem is also impossible.

[10]Two cells containing a vertex A are *vertical with respect to* A if their angles A are vertical.