Discharging method*

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1 Double counting

1.1. Some cells of a given table are marked. For every marked cell, the number of marked cells in its column equals to the number of the marked cells in its row. Prove that the number of rows with at least one marked cell is equal to the number of columns with at least one marked cell.

Solution. Let us put in every marked cell c the number $\frac{1}{k}$, where k is the number of marked cells in the column (the row) of c. Clearly, the sum of numbers in each non-empty row is 1, and so the total sum of all numbers in the table equals the number of non-empty rows. Analogously the total sum of all numbers in the table equals the number of non-empty columns.

1.2. E elves and D dwarfs arrived to the Zilantkon convention. It turns out that during the convention every dwarf got into a fight with at least one elf and every elf got into a fight with at most ten dwarfs. Also, it is known that every dwarf has more opponents-elves than any of his opponents-elves has opponents-dwarfs. Prove that $D \leq \frac{10}{11}E$.

Solution. Let us asing charge 1 to every elf, then the total charge is equal to E. If an elf has k opponents-dwarfs, then he gives the charge $\frac{1}{k}$ to each of those.

If an dwarf Duda has m opponents-elfs, then each of those has at most $m-1$ opponents-dwarf, where $m \leq 11$, and thus give Duda a charge at least $\frac{1}{m-1}$. Therefore, each dwarf obtains charge at least $\frac{m}{m-1} \geq \frac{11}{10}$.

Thus, the final charge of dwarfs is at least $\frac{11}{10}D$. Hence, $\frac{11}{10}D \leq E$.

1.3. A table has m rows and n columns, where $m < n$. Some cells are marked in such a way that every column contains at least one marked cell. Prove that there is a marked cell such that the number of marked cells in its row is larger than the number of marked cells in its column.

Solution. Let us give to every column a charge (-1) and to every row a charge 1. Clearly, the total charge is negative. Let every row or column distribute its charge evenly to each of its marked cells. Since the total charge is negative there is a cell with a negative charge. It is easy to see that this is a desired cell.

1.4. In a school library there were exactly k empty bookshelves yesterday. Today in the morning, some books were rearranged in such a way that every shelf is not empty anymore. A book is called *boring* if the number of books on its current shelf is less than

*Here you can find the current results: https://clck.ru/HMnm8 the number of books on its shelf before the rearrangement. Prove that there are at least $k+1$ boring books.

Solution. In the solution of this problem we are interested only in the change of charges of books.

First, if a book initially lied on a shelf with n other books, then it take a charge $\frac{1}{n}$ from its shelf. If a book after rearrangement lies on a shelf with m other books, then it gives a charge $\frac{1}{m}$ to its shelf.

Then the total change of charges of books decreases by k. But the change of a charge of one book is less than 1. Therefore, there are $k+1$ desired books.

1.5. Suppose that an $n \times n$ table is filled **a)** with numbers 0 and 1 **b)** with non-negative integers in such a way that if some cell of the table contains 0, then the sum of all numbers in its cross[1] is at least 1000. Find the least possible sum of numbers in the table.

Solution.

1.6. Suppose that a convex n-gon and m red points distinct from the vertices of the polygon are drawn on a blackboard. It turns out that each segment between two vertices of the polygon contains at least one red point. Prove the inequality

$$m \geq n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\lfloor (n-1)/2 \rfloor}\right).$$

Solution. See the proof of Theorem 3 in [Mat09].

1.7. Consider n unit circles drawn in the plane. It is known that each of them intersects with at least one another circle and also there are no two touching circles. It is possible that more than two circles pass through one point. Prove that there are at least n intersection points.

Solution. See the proof of Theorem 1 in [LP07].

1.8. A square is cut into several triangles. Prove that there are two triangles sharing a common edge.

Solution. See the proof of Theorem 6 in [KPT17].

1.9*. There are n lines in general position in the plane, that is, no three of them share a common point and no two are parallel. These lines split the plane into several parts. Prove that there are at least $n-2$ triangles among them.

Solution. See the proof of Theorem 5.15 in [Fel12].

2 Problems about graphs

Planar graphs

Here, we present the common idea of the proofs given in this subsection.

For a connected planar graph Γ, consider Euler's Formula

$$V(\Gamma) - E(\Gamma) + F(\Gamma) = 2.$$

[1] A cross of a cell is the union of cells lying with it in the same row or the same column

Multiplying the equality by (-2) and using $2E(\Gamma) = \sum_{v\in V} \deg v = \sum_{f\in F} \deg f$, we can rewrite Euler's Formula in the following way

$$\sum_{v\in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f\in F(\Gamma)} ((1-\alpha)\deg f - 2) = -4. \tag{1}$$

Choosing different values of α, we obtain different equalities. Let us assign charges to the vertices and the faces of Γ according to the left-hand side of (1). Of course, the total charge is negative. Then we suppose the contrary to the desired statement of the problem. Using that and the properties of the graph given in a problem, we rearrange the charges in a way that the final charge of every element (a vertex or a face) becomes non-negative. Usually, this contradiction finishes the proof.

2.1. Let V be a convex polyhedron without 4-gonal and 5-gonal faces. Prove that V has at least 4 triangular faces.

Solution. Let us project the polyhedron on the plane in a way we obtain a plane graph. Choosing $\alpha = \frac{2}{3}$ in (1) and multiplying it by 3, we have

$$\sum_{v\in V}(2\deg v - 6) + \sum_{f\in F}(\deg f - 6) = -12.$$

Let us assign charges in the following way: a vertex v obtains a charge $(2\deg v - 6)$ and a face f gets a charge $(\deg f - 6)$. Since every vertex has degree at least 3, its charge is non-negative. If $\deg f = 3$, then its charges is (-3); if $\deg f \geqslant 6$, then its charge is non-negative. Since the total charge equals (-12), then there are 4 at least 4 faces with the charge (-3), that is, 4 triangular faces.

Note that in this proof we did not rearrange the charges between faces and vertices.

2.2. Let Γ be a planar graph with $\delta(\Gamma) \geq 2$ and all cycles of length at least 7. Prove that there is an edge of Γ of weight at most 5.

Solution. See the proof of Lemma 6.11 in [CW13].

2.3. Let Γ be a planar graph with $\delta(\Gamma) \geq 5$ such that all its faces are triangles, and there is no two adjacent vertices of degree 5. Prove that there is a face with degrees of vertices 5, 6 and 6, respectively.

Solution. Consider (1) for $\alpha = \frac{1}{2}$ and multiply by 2. Thus, we obtain

$$\sum_{v\in V}(\deg v - 4) + \sum_{f\in F}(\deg f - 4) = -8.$$

Assign to a vertex v the charge $\deg v - 4$ and to a face f a charge $\deg f - 4$.

Then every vertex of degree 5 gives to each of its faces a charge $\frac{1}{5}$, every vertex 6 gives charge $\frac{1}{3}$ and every vertex of degree at least gives charge $\frac{2}{5}$.

Then the charges of vertices of degree 5 or 6 became equal to 0. For vertices of degree n, where $n \geq 7$, the charge becomes equal to $n - 4 - \frac{2}{5}n = \frac{3}{5}n - 4 \geqslant \frac{3}{5} \cdot 7 - 4 > 0$.

Initially every vertex of degree at least 4 was non-negative and after rearrangement of charges it is still non-negative. Now consider a triangular face. Suppose the contrary the charge of every edge is at least 12. Then the set of degrees of every face must be

(5, 7+, 7+) or (6+, 6+, 6+)[2]. It is easy to see in both cases the charge of every face is non-negative.

2.4. Let Γ be a planar graph with $\delta(\Gamma) \geq 5$. Prove that there is an edge of weight at most 11.

Solution. See the subsection 'An easy example' on the webpage https://en.wikipedia.org/wiki/Discharging_method_(discrete_mathematics)

2.5. Let Γ be a planar graph such that $\delta(\Gamma) \geq 3$. Prove that there are a face f and a vertex $v \in f$ such that either $\deg(v) = 3$ and $\deg(f) \leq 5$ or $\deg(v) \leq 5$ and $\deg(f) = 3$.

Solution.

Multiplying the equality (1) for $\alpha = \frac{1}{2}$ and multiply, we have

$$\sum_{v \in V}(\deg v - 4) + \sum_{f \in F}(\deg f - 4) = -8.$$

Assign to a vertex v a charge $\deg v - 4$ and to a face f a charge $\deg f - 4$.

Suppose the contrary that every face of degree 3 does not have vertices of degree less than 6 and every vertex of degree 3 is not incident to a face of degree at less than 6.

Then every vertex of degree at least 6 gives to every incident face a charge $\frac{1}{3}$. Therefore, the charge of such a vertex is non-negative and every triangular face has charge equal to 0.

Also, every face of degree at least 6 gives to its vertices the charge $\frac{1}{3}$. Therefore, the charges of such faces are non-negative and every triangular face has a charge equal to 0. A contradiction.

2.6. Let Γ be a connected planar graph such that $\delta(\Gamma) \geq 3$. Prove that there is a face of length at most 5 with the degrees of all its vertices but one do not exceed 11.

Solution. Multiplying the equality (1) for $\alpha = \frac{2}{3}$ by 3, we have

$$\sum_{v \in V(\Gamma)} (2\deg v - 6) + \sum_{f \in F(\Gamma)} (\deg f - 6) = -12.$$

Assign to a vertex v a charge $2\deg v - 6$ and to a face f charge $\deg f - 6$.

Suppose the contrary, that is, every face of degree at most 5 has at least two vertices of degree at least 12.

Let every vertex of degree n, where $n \geq 12$, give to every incident face a charge $\frac{3}{2}$. Then its charge becomes equal to $2n - 6 - \frac{3}{2}n = \frac{1}{2}n - 6 \geq 0$. Every face of degree at most 5 will obtain charge from at least two vertices, and so its charge is at least $-3 + \frac{3}{2} \cdot 2 = 0$. All the rest vertices and faces initially had non-negative charges and these charges do not decrease.

Light colorings

2.7. a) Prove that any planar graph is 6-colorable.

b) Suppose that, for any subgraph of a given graph Γ, one of the following conditions holds: 1) there is a vertex of degree at most $d - 1$; 2) there is an even induced cycle such

[2] Here, we denote by $k+$ a number that is at least k.

that the degree of every its vertex does not exceed d. Prove that the induced graph Γ is d-colorable.

Solution. **a)** Every face of a planar graph Γ has at least 3 edges, thus, $3|F(\Gamma)| \leq 2|E(\Gamma)|$. Substituting this inequality in Euler's Formula, we obtain

$$6 = 3|V(\Gamma)| - 3|E(\Gamma)| + 3|F(\Gamma)| \leq 3|V(\Gamma)| - |E(\Gamma)|.$$

It means $3V > E$, and so there is a vertex of degree at less than 6.

Now, it is easy to finish the proof using induction. We leave it as an exercise for the reader.

b) Prove this statement for all subgraphs of Γ by induction on the number n of vertices.

If in Γ there is a vertex v of degree at most $d-1$, then color $\Gamma \setminus \{v\}$ in d colors (by induction) and then choose the proper color for v.

Otherwise, in Γ there is an even induced cycle $v_1 \dots v_k$ such that the degree of each vertex is at most d. Denote by C the set of its vertices. By the induction hypothesis, we color $\Gamma \setminus C$ in a proper way. Now, let us show how we can color vertices of C. For every vertex in C, there is at most $d-2$ edges connecting it with vertices in $\Gamma \setminus C$, and thus for each vertex in C there are at least 2 possible colors.

Let us show the following Lemma.

Lemma. If, to every vertex of an even cycle, a list of 2 colors is assigned, then it is possible to color it in proper way choosing colors from the corresponding list.

If lists of the vertices of the cycle are the same, then, clearly, it is possible to color it alternatively. Otherwise, there are two neighbouring vertices with distinct lists, say, the vertices v_1 и v_k. Color v_1 in a color that is not in the list of v_k. Then there is a choice for coloring v_2, and so we can color it. Analogously, we can color $v_3, \dots, v_k$. Thus, all vertices are colored in the proper way, including v_1 and v_k.

2.8. a) Prove that for a given k there exists an integer n such that the complete bipartite graph $K_{n,n}$ is not k-choosable. Although it is 2-colorable as any bipartite graph.

b) Prove that any even cycle is not only 2-colorable but also 2-choosable.

Solution. **a)** Let the first part of the graph has vertices, indexed by numbers $1, \dots, k$. Also, assume that the vertex i has list $(c_{i1}, c_{i2}, \dots, c_{ik})$, here we assume that all these k^2 colors are distinct. In the second part there are k^k vertices with all possible lists of the form $(c_{1x_1}, c_{2x_2}, \dots, c_{kx_k})$, where $x_i \in \{1, \dots, k\}$ for all i.

Then for any coloring of the first part there is a vertex of the second part, whose list coinside with the coloring of the first part, and therefore it is impossible to color it properly. Since we are going to construct an example with the same number of vertices in both parts, we add extra vertices to the first part (with arbitrary lists).

b) See the lemma in the proof of Problem 2.7b.

2.9. a) Suppose that every inner face of a planar graph Γ is a triangle and its outer face is bounded by a cycle $v_1 \dots v_k v_1$. Suppose further that with v_1 and v_2 lists of 2 colors are associated, with every other vertex of the outer face a list of 3 colours, and with every inner vertex a list of 5 colours. Then there exists a proper list coloring of Γ for the given lists.

b) Every planar graph is 5-choosable.

Solution. See the proof of Theorem 5.4.2 in [Die12].

2.10. Consider a planar graph Γ with $\Delta(\Gamma) \geq 11$. Assign lists of size $\Delta(\Gamma)$ to the vertices of Γ and lists of size $\Delta(\Gamma)+2$ to the edges. Prove that Γ has a proper total list coloring for this set of lists.

Solution. See the proof of Theorem 6.3 (and the proof of Lemma 6.1) in [CW13].

3 Main problems

3.1 Matchstick graphs

Key Problem 1. Prove that a 5-regular matchstick graph does not exist.

Note. Actually, in any matchstick graph, there is a vertex with degree less than 5.

3.1.1. Let Γ be a graph of minimal distances on n vertices that are in general position[3]. Prove that **a)** $E(\Gamma) < 5n/2$, **b)** there exists a constant $c < 5/2$ such that $E(\Gamma) \leq cn$.

c) An *interesting graph* is a graph of minimal distances such that, for every vertex, it and its neighbors are in general position[4]. Prove that, for every $c < 5/2$, there is an interesting graph Γ with at least $c|V(\Gamma)|$ edges.

Solution. See [Tót97].

3.1.2. Solve Key Problem 1.

Solution. See [KP11].

3.1.3*. Prove that a 4-regular matchstick graph contains at least 20 vertices.

Solution. See Section 3 and Theorem 3.6 in [KP14].

3.2 Quasi-planar graphs

Key Problem 2. a) Prove that $|E(\Gamma)| \leq 8n - 20$ for a quasi-planar graph Γ on n vertices.

b) Try to improve the statement of the previous problem. Both sharpening of the inequality and its generalizations for other classes of graphs are interesting.

Solution. See [AT07].

3.3 List edge coloring

Key Problem 3.

Solution. See the proof of Theorem 12 in [Cra05].

3.4 Magical configurations

Key Problem 4. Describe all magical configurations.

Solution. See [ABK+09].

[3]that is, no three of them are on the same line

[4]all vertices are not necessary in general position

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