# Sages and hats

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## 1 Get acquaintance: “Hats” game

Let undirected graph $G$ be given, there is one sage and one chest with hats of different colors in each vertex of $G$. The sages know each other. Graph $G$, location of the sages in graph vertices and colors of hats in the chests are fixed and known to all. In particular, each sage understands, in which vertex each of his neighbours is situated. Judge conducts the following test with the sages.

Judge places a hat on head of each sage from the corresponding chest. Each sage sees only the hats of sages in neighbour graph vertices, but he does not see his own hat and does not know its color. The sages are not allowed to communicate. By judge signal the sages name a color simultaneously. We say that the sages have passed the test successfully = «have win», if at least one of them has named correct color of his own hat.

Rules were explained to the sages before the test and they had possibility to hold a discussion, in which they have to determine a public strategy. Publicity means that everybody, including the judge, knows this strategy. The strategy of sages should be deterministic, i.e. each sage decision is determined uniquely by the hats of other sages. We call a strategy *winning* if for any hat placement at least one sage names correctly color of his own hat. Also we say that the sages win if they have a winning strategy and loose if they have not.

Here is a pair of problems on this topic.

**1.1.** Each of two sages obtains a hat of white, blue or red color, placed on his head. The sages received the message before the test that the colors of their hats are different. Each of them sees the hat of the other sage but does not see his own hat. They must try to guess simultaneously colors of their hats (each of them writes a color on piece of paper). Prove that the sages can come to agreement how to act in advance so that at least one of them will guess the color correctly.

**1.2.** Sultan gives examination for 11 court sages (viziers). By examination rules sultan puts 10 sages on 10 pits, located around a circle, and he imprisons one more sage to the tower in the centre of the circle. On forehead of each of the first 10 sages sultan writes number 1 or 2; on forehead of the central sage sultan writes a number from 1 to 1024. The sage in the tower sees numbers of all the others, and they see his number (but do not see each other). All the sages must try to guess their own numbers simultaneously. Sultan explained to the sages the rules of examination in advance and gave them time for discussion before the examination starts. Can the sages act so that at least one of them certainly guesses his number?

We will identify the vertex of graph $G$ and the sage in it. We mean that colors are numbered by 0, 1, 2, 3, …, and that hats with colors from 0 to some number $h(X)-1$ lie in the chest of sage $X$, the set of hat colors in the chest of sage $X$ is denoted by $\operatorname{Col} X$.

We call a *hats game* a pair $\langle G, h\rangle$, where $G=\langle V,E\rangle$ is a graph, $h\colon V\to\mathbb{N}$ is a function that shows how many hats are in the chest in each vertex. We call function $h$ «hatness». We will sometimes write $\widehat{A}$ instead of $h(A)$. If the sages win in game $\langle G, h\rangle$, we say that $\langle G, h\rangle$ is a *winning graph*.

We call a winning graph *simple*, if it does not contain winning subgraph $\langle G', h'\rangle$, where $G'\subsetneq G$, $h'=h\big|_{V(G')}$.

**1.3.** Prove that the sages win on the graph in fig. 1.

**1.4.** Is winning graph in fig. 1 simple?

**1.5.** Prove that the sages win on the graph «path $A_1A_2\dots A_n$», where $n\geqslant 2$ and hatnesses are given in fig. 2.

**1.6.** Is winning graph in fig. 2 simple?

**1.7.** Prove that the sages win on graph $K_{2,3}$ with given hatnesses (fig. 3).

**1.8.** Do the sages win on graph $K_{1,3}$ with given hatnesses (fig. 4)?

**1.9.** Prove that the sages win on the cycle $C_n$, $n\geqslant 4$, with given hatnesses (fig. 5).

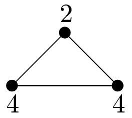



Figure 1.


Figure 2. Path with $n$ vertices

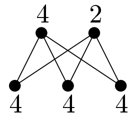



Figure 3. Graph $K_{2,3}$

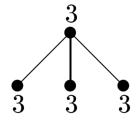



Figure 4. Graph $K_{1,3}$

**1.10.** Prove that the sages win on the cycle $C_n$, $n \geqslant 4$ with given hatnesses (fig. 6).

**1.11.** Does there exist a simple winning graph, on which at least two sages guess their own colors correctly for any hats placement?

**1.12.** Prove that on complete bipartite graph $K_{99,50}$ sages loose, if hatness of each sage equals 100.

**1.13.** Sages are on the graph «path $AB$» (fig. 7, $n \geqslant 4$). Before the discussion the judge promised to the sages that $A$ and $B$ will get hats of the same color. Do the sages win?

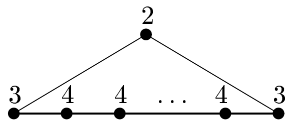



Figure 5. Cycle with $n$ vertices

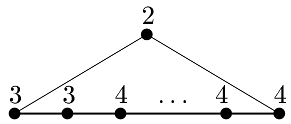



Figure 6. Cycle with $n$ vertices

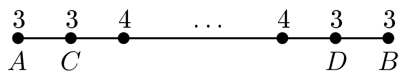



Figure 7. Path $AB$ with $n$ vertices

**1.14.** Let sages play on graph $G$ with hatness function $h$. The judge promised them in advance that during the test when he will put a hat on the sage $A$ head he will whisper into his ear a true statement of the form «I has placed on your head a hat of one of two colors $c_1$ or $c_2$». Therefore, in the discussion the sages know that during the test the judge will give to a sage $A$ a hint, but they do not know what colors he will name. So the sages determine strategies for all the sages except $A$, as usual, and sage $A$ gets the set of $\binom{h(A)}{2}$ strategies, one for each possible judge's hint.

Prove that this hint does not affect the game result.

## 2 Games on cliques

**2.1.** Does there exist a winning graph not containing a 4-clique as subgraph, for which hatnesses of all sages equal 4?

**2.2.** Sages are in vertices of complete graph $K_n$, in the chest of $i$-th sage there are $a_i$ hats of different colors. Prove that the sages win if and only if

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geqslant 1.$$

**2.3.** We call a strategy of sages *exact*, if for each hats placement exactly one sage guesses his color correctly. Find all games for which an exact strategy exists.

**2.4.** Graph $G$ is obtained from complete graph on $n$ vertices by removing one edge. The hatnesses of all sages equal $n$. Do the sages win on such graph?

**2.5.** Graph $G$ is complete graph with 4 vertices $A$, $B$, $C$, $D$, in which edge $CD$ has been removed. And $h(A) = 6$, $h(B) = 6$, $h(C) = 2$, $h(D) = 3$. Do the sages win on such graph?

**2.6.** Graph $G$ is complete graph with $n$ vertices $A_1$, $A_2$, …, $A_n$, in which edge $A_{n-1}A_n$ has been removed. The hatnesses of vertices equal $a_1$, …, $a_n$. It turned out that the graph is winning and

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} - \frac{1}{a_{n-1}a_n} = 1.$$

Prove that $a_1 a_2 \ldots a_{n-2}$ is divisible by $a_{n-1}a_n$.

**2.7.** Do the sages win on the graph «Medium bow» (fig. 8)?

**2.8.** Do the sages win on the graph «Big bow» (fig. 9)?

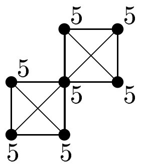



Figure 8. Graph «Medium bow»

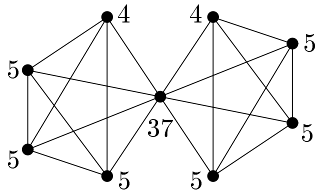



Figure 9. Graph «Big bow»

## 3 Graph operations

Let $\langle G_1, h_1\rangle, \langle G_2, h_2\rangle$ be two games such that $V_1 \cap V_2 = \{v\}$. Let $G = G_1 \underset{v}{+} G_2$ be the union of graphs $G_1$ and $G_2$, in which both vertices $v$ are united into one vertex. Let function $h\colon V_1 \cup V_2 \to \mathbb{N}$ coincide with $h_i$ on $V_i \setminus \{v\}$ $(i = 1, 2)$ and $h(v) = h_1(v)h_2(v)$. Denote the game $\langle G, h\rangle$ by $G_1 \times_v G_2$ (fig. 10).

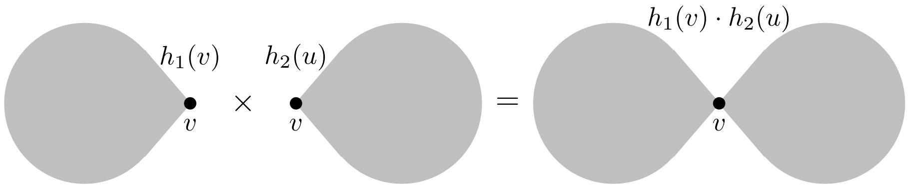



Figure 10. Game $G_1 \times_v G_2$

**3.1.** Theorem about product. If sages win on graphs $G_1$ and $G_2$, then they win also on graph $G_1 \times_v G_2$.

Let $G_1$ and $G_2$ be two graphs without common vertices. By the *substitution of graph $G_2$ to graph $G_1$ on the place of vertex $v$* we call the graph obtained by union of graphs $G_1 \setminus v$ and $G_2$ with adding of all edges, that connect each vertex of $G_2$ with each neighbour of $v$. We denote the substitution by $G_1[v := G_2]$.

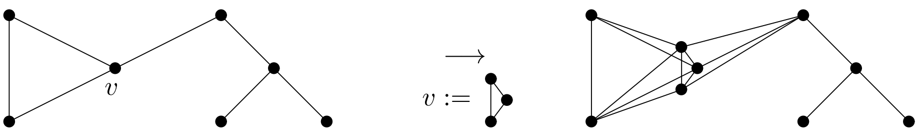



Figure 11. Substitution of graph on the place of vertex

**3.2.** Theorem about substitution. Let sages win in games $\langle G_1, h_1\rangle$ and $\langle G_2, h_2\rangle$. Let $G$ be the graph of substitution $G_1[v := G_2]$, where $v \in G_1$ is an arbitrary vertex. Then game $\langle G, h\rangle$ is winning, where

$$h(u) = \begin{cases} h_1(u) & u \in G_1, \\ h_2(u) \cdot h_1(v) & u \in G_2. \end{cases}$$

**3.3.** Let $\langle G, h\rangle$ be a game, in which sages win, let $BC$ be an edge of graph $G$. Consider graph $G' = \langle V', E'\rangle$, obtained by adding new vertex $A$ to graph $G$: $V' = V \cup \{A\}$, and two edges: $E' = E \cup \{AB, AC\}$. Then the sages win in the game $\langle G', h'\rangle$ (see fig. 12), where

$$h'(u) = \begin{cases} 2, & u = A, \\ 2h(B), & u = B, \\ 2h(C), & u = C, \\ h(u), & \text{for other vertices } u \in V. \end{cases}$$

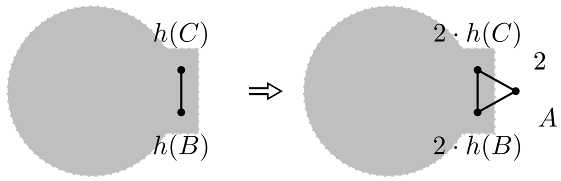



Figure 12. Addition of vertex $A$ of hatness 2 to edge $BC$

**3.4.** Let $\langle G_1, h_1\rangle$, $\langle G_2, h_2\rangle$ be two games, in which sages win. Let $A_1$, $A_2$, …, $A_k \in V_1$; $B_1$, $B_2$, …, $B_m \in V_2$. Consider graph $G' = \langle V', E'\rangle$, obtained by adding all the edges $A_iB_j$ to graph $G_1 \cup G_2$: $V' = V_1 \cup V_2$, $E' = E_1 \cup E_2 \cup \{A_iB_j, i = 1, \dots, k; j = 1, \dots, m\}$ (fig. 13). Then the sages win in game $\langle G', h'\rangle$, where

$$h'(u) = \begin{cases} h_1(u), & u \in G_1 \setminus \{A_1, A_2, \dots, A_k\}, \\ h_2(u), & u \in G_2 \setminus \{B_1, B_2, \dots, B_m\}, \\ h_1(u) + 1, & u \in \{A_1, A_2, \dots, A_k\}, \\ h_2(u) + 1, & u \in \{B_1, B_2, \dots, B_m\}. \end{cases}$$

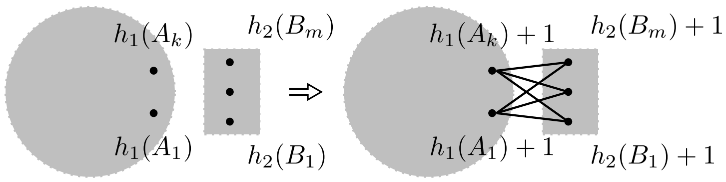



Figure 13. Gluing of two graphs, $k = 2$, $m = 3$

**3.5.** Let $\langle G, h\rangle$ be a game in which sages win, let $Z$, $C \in V$ be two vertices of graph $G$. Consider graph $G' = \langle V', E'\rangle$, obtained by adding new path $ZABC$ to graph $G$: $V' = V \cup \{A, B\}$, $E' = E \cup \{ZA, AB, BC\}$. Then the sages win in game $\langle G', h'\rangle$, where $h'(Z) = 2h(Z)$, $h'(C) = h(C) + 1$, $h'(A) = 2$ and $h'(B) = 3$ and $h'(u) = h(u)$ for other vertices $u \in V$ (fig. 14).

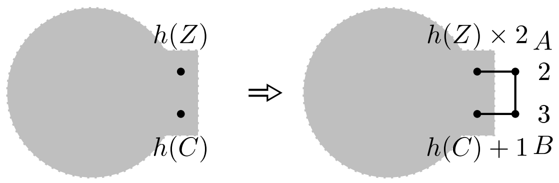



Figure 14. Addition of two vertices with hatnesses 2 and 3 connected by edge

**3.6.** Let graph $\langle G, h'\rangle$ be winning, $V(G) = \{A_1, \dots, A_n\}$. Take $n$ winning graphs $\langle G_i, h_i\rangle$ and mark in each of them one vertex $A_i$. Construct graph $G$ on the set of marked vertices. For this «supergraph» define hatness function as $h(A_i) = h'(A_i)h_i(A_i)$ (and for the other vertices hatness is the same as in the initial graphs). Then the sages win.

**3.7.** The addition to graph $G$ a pendant vertex $A$, where $\widehat{A} > 2$, does not affect the result of the game.

**3.8.** a) Addition of two new vertices $B$ and $C$ of hatness 5 and edges $AB$, $AC$, $BC$ to graph $G$ with vertex $A$ of hatness 2 does not affect the result of the game.

b) Addition of two new vertices $B$ and $C$ of hatnesses 7 and 5 and edges $AB$, $AC$, $BC$ to graph $G$ with vertex $A$ of hatness 3 does not affect the result of the game.

**3.9.** Let $G_1$ and $G_2$ be graphs with vertex $A$ in which sages loose if $\widehat{A} = 2$. Then the sages loose on graph $G = G_1 +_A G_2$, in which $\widehat{A} = 2$.

**3.10.** Let $H = \langle G, h\rangle$ be a loosing game, $B$ be an arbitrary vertex of graph $G$. Consider graph $G' = \langle V', E'\rangle$, obtained by addition new pendant vertex $A$ to graph $G$: $V' = V \cup \{A\}$, $E' = E \cup \{AB\}$. Then the sages loose in game $\langle G', h'\rangle$, where $h(A) = 2$, $h'(B) = 2h(B) - 1$ and $h'(u) = h(u)$ for other vertices $u \in V$.

## 4 Blind chess

**Game «*Check by rook*».** Two chess players $\mathcal{L}$ and $\mathcal{R}$ are sitting opposite each other, there is a chessboard on the wall behind of each of them. Each chess player does not see his own board but sees the board of the other chess player. The judge puts one king on each of these boards. After that the chess players must to name one cell of his chessboard independently of each other and the judge puts rook on this square. If at least one of the kings is in rook check (under attack of rook) (or the rook has been put on the same cell, where king is), then the chess players win, otherwise they lose.

Chessboards of the players can be different and of arbitrary sizes. As in the game «Hats», the chess players determine public deterministic strategy in advance. The judge knows this strategy and plays against the chess players.

Let graph $G$ be the cycle $ABCDA$ with hatness function $h$. In fact, graph $G$ is the complete bipartite graph $K_{2,2}$, so sages $A$ and $C$ see the same, and sages $B$ and $D$ also see the same. Then a pair of players $A$ and $C$ we call chess player $\mathcal{L}$ and we will mean that his board has size $h(A) \times h(C)$, and a pair $B$ and $D$ we call chess player $\mathcal{R}$ and we will mean that his board has size $h(B) \times h(D)$. Therefore game «Hats» on the cycle $ABCDA$ with hatness function $h$ is equivalent to game «Rook check» on boards $L(h(A) \times h(C))$ and $R(h(B) \times h(D))$.

Union of any vertical and any horizontal row of chessboard we call *cross*. Cross is uniquely defined by the cell, situated in the intersection of these rows, it is called the *centre* of the cross. The rook located in the centre of the cross, holds in check exactly all cells of the cross.

In problem 4.1 you can submit items solutions separetaly.

**4.1.** In the game «Check by rook» the chess players win on the following pairs of board:

W1) one of the boards has size $1 \times k$, where $k$ is an arbitrary positive integer;
W2) $L(2 \times k)$ and $R(2 \times m)$, where $k$ and $m$ are arbitrary positive integers;
W3) $L(3 \times 3)$, $R(3 \times 3)$; W4) $L(2 \times 3)$, $R(3 \times 4)$;
W5) $L(2 \times 4)$, $R(3 \times 3)$; W6) $L(2 \times 2)$, $R(k \times m)$, where $\min(k, m) \leqslant 4$.

The chess players lose on the following pairs of boards:

L1) $L(2 \times 3)$, $R(4 \times 4)$; L2) $L(2 \times 3)$, $R(3 \times 5)$;
L3) $L(2 \times 4)$, $R(3 \times 4)$; L4) $L(3 \times 3)$, $R(3 \times 4)$;
L5) $L(2 \times 2)$, $R(5 \times 5)$; L6) $L(2 \times 5)$, $R(3 \times 3)$.

For boards of other sizes the question if the sages win can be solved by comparing with these cases. For example, the chess players lose on the boards $L(3 \times 4)$, $R(3 \times 4)$ because they lose even in simpler case L3. The chess players win on the boards $L(2 \times 3)$, $R(3 \times 3)$ because they win even when one of the boards is larger (as in case W3).

**Check by queen.** Consider variant of the game when both chess players put on the board a queen instead of a rook. We call this game «Check by queen».

**4.2.** In the game «Check by queen» $L(4 \times 5)$, $R(4 \times 5)$ the chess players win.

**4.3.** In the game «Check by queen» $L(4 \times 4)$, $R(5 \times 5)$ the chess players win.

**4.4.** In the game «Check by queen» $L(7 \times 8)$, $R(7 \times 7)$ the chess players lose.

**4.5.** In the game «Check by queen» $L(3 \times 4)$, $R(7 \times 7)$ the chess players lose.

**4.6.** In the game «Check by queen» $L(4 \times 5)$, $R(5 \times 5)$ the chess players lose.

**4.7.** Consider variant of the game «Check by queen», in which 5 chess players are situated so that each of them sees the boards of the others but does not see his own board. All boards have size $11 \times 11$ As in the initial game, the judge puts one king on each board, and the chess players simultaneously point to the cell, where the queen has to be put. Can the chess players win?

## 5 Several more problems

**5.1.** Let $G$ be graph with vertices $B$ and $C$. Let hatness function is such that $\widehat{B} = \widehat{C} = 2$ and the graph is losing. Add to the graph new vertex $A$, which is connected only with $B$ and $C$. Then the sages lose on the obtained graph, if $\widehat{A} = 2$, $\widehat{B} = 3$, $\widehat{C} = 7$, and hatnesses of other vertices have not changed.

**5.2.** Is it true that there exists such «large» number $k$ that any graph $G$, in which degrees of all vertices do not exceed 3, and hatnesses of all vertices equal $k$, is losing?

**5.3.** Is it true that there exists such «large» number $d$, that any graph $G$, in which degrees of all vertices equal $d$, and hatnesses of all vertices equal 4, is winning?

# SOLUTIONS

**1.1.** They can arrange colors on cycle, for example, white→blue→red→white, and further let each of them will write next color in order of the cycle, looking at the color of fellow's hat.

**1.2.** Answer: yes.

Let the number on the tower be written in binary notation, and the numbers 0 or 1 be written on the forehead of each of the sages in pits (instead of 1 and 2). Let the $k$-th sage in the pit play according the strategy «my bit does not coincide with the $k$-th bit on the tower», and the sage in the tower only compose his number from the bits in pits.

**1.3.** Let colors be numbers from 0 to 3. Let the upper sage call the parity of sum of hat colors on two other sages. And the other two sages, since they see the third one and each other, can understand what two colors they must exclude, and play as if they are on the graph-edge with two colors.

**1.4.** Answer: yes.

It is enough to check that after removing of one edge losing graph is obtained, i.e. that graph «path with two edges» with hatnesses 2, 4, 4 or 4, 2, 4 is losing.

Apply probabilistic arguments: so, for the case 2, 4, 4, the fractions of the number of placements, for which the sages win, equals 1/2, 1/4, 1/4 correspondingly, that is exactly 1 in total. But the cases, in which the sages in outermost vertices guess correctly, are independent: on 1/8 part of all placements both outermost sages guess correctly. Therefore, the total part of all placements, on which the sages guess correctly, equals $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}-\frac{1}{8}<1$.

Another case is considered similarly.

**1.5.** It is well known that the sages win for $n=2$, when hats can be one of two colors: the first plays according the hypothesis «we have equal hats», and the second «we have different hats». Applying the problem 3.1 several times, we obtain the statement for the other $n$.

**1.6.** Answer: yes.

It is enough to check that graph «path $P_k$» with hatnesses $2, 4, \ldots, 4$ is losing. Using the constructor from problem 3.7 this statement is easily reduced to the checking whether the game of two persons with hatnesses 2 and 4 is wining or not.

But it can be done also by induction. Let $A$ be the sage with hatness 2, $B$ be his neighbour. By the strategy sage $A$ depending on the color of $B$ (four variants) says one of two colors, say, red or blue. Let sage $A$ say blue color in two or more cases («bad» placements), and say red in the other cases («good» placements). Let the judge give red hat to sage $A$ and declares that sage $B$ will get hats of some two colors corresponding to bad placement. Then sage $A$ will certainly guess incorrectly, and sage $B$ can consider that his hatness equals 2. The game is reduced to shorter path of the same kind.

**1.7.** Let $X$ and $Y$ be two vertices in the first part, and $A$, $B$, $C$ be vertices in the second part, and $\widehat{Y}=2$, $\widehat{X}=\widehat{A}=\widehat{B}=\widehat{C}=4$. Describe the winning strategy of the sages. Each of four colors can be interpreted as 2-bit binary number consisting of the *left* and the *right* bits. If a sage is ready to say color of defined parity, he already determined with right bit and must to choose one of two values of left bit. The color of $Y$'s hat, one bit, we also will interpret as parity.

Let sage $Y$ say the parity $y$ that is dominant among hats $A$, $B$, $C$.

Each of sages $A$, $B$, $C$ sees hat of $Y$. So they understand that if the most of hats $A$, $B$, $C$ have the same parity as $Y$'s hat, then sage $Y$ guesses correctly color of his hat. Therefore, they act assuming that the most of hats has parity $\overline{Y}$. Hence, in order to say their guesses each of them must choose a color between two possible colors with parity $\overline{Y}$.

From the other hand, sage $X$ ( and sage $Y$ too) sees, what parity is dominant among hats $A$, $B$, $C$, and knows the value of $y$. Therefore, sage $X$ knows that if the hat of color $y$ is on sage $Y$, then $Y$ has guessed correctly. Hence sage $X$ assumes that hat of color $\overline{y}$ is on sage $Y$. But then he knows that in this case sages $A$, $B$, $C$ proceed from the assumption that most of their hats have parity $y$. Thus, sage $X$ sees what is dominant parity among colors $A$, $B$, $C$, and he knows that sages $A$, $B$, $C$ proceed from the assumption that this parity is really dominant among their hats. Let sage $X$ choose the two sages with hats of dominant parity among $A$, $B$, $C$, we call these sages *main*.

Now we show how the «assumptions» can be transformed into a strategy. Columns of the following table correspond to colors of hat $X$, and rows correspond to sages $A$, $B$, $C$. Let each of sages $A$, $B$, $C$ take left bit from his own row of table for his guess (and right bit is defined by the «assumption»).

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $A$ | 0 | 0 | 1 | 1 |
| $B$ | 0 | 1 | 0 | 1 |
| $C$ | 0 | 1 | 1 | 0 |

Sage $X$ sees whether somebody of main sages among $A$, $B$, $C$ has guessed correctly the color of his own hat by the given strategy or not. If two main sages have guessed incorrectly then it means that they both pointed wrong left bit of color of their own hats.

Let, for example, main sages be $A$ and $B$, let they point to bits 0 and 1 correspondingly and have not guessed correctly. Sage $A$ points to bit 0, only if the color of hat $X$ equals 0 or 1; sage $B$ points to bit 1, only if the color of hat $X$ equals 1 or 3. Therefore, both events happen only in the case when the color of hat $X$ equals 1, and sage $X$ must say exactly this color.

Sage $X$ performs similar actions in all other cases. His success is guaranteed by the following *magic property* of the table.

*If two arbitrary rows are chosen in the table and two cells in each of them (independently of the other), containing equal symbols are painted over , then exactly one column contains two painted over cells.*

Magic property is checked by (evident) considering of all the possible cases. Note that weaker requirement «no more than one column of the table contains two painted over cells» is enough to provide the strategy to be winning.

**1.8.** Suppose that the sages have a winning strategy. Let $v$ be the vertex of degree 3, $u_1$, $u_2$, $u_3$ be pendant vertices. Will assign the first color to vertex $v$. Let sages $u_1$, $u_2$, $u_3$ say colors $h_1$, $h_2$, $h_3$ according to the strategy.

Now conduct the second experiment: assign second color to vertex $v$. Let sages $u_1$, $u_2$, $u_3$ say colors $e_1$, $e_2$, $e_3$ according to the strategy.

Finally, conduct the last experiment. For each $i = 1, 2, 3$ denote by $d_i$ the color that has not been said by sage $u_i$ in the first two experiments (if we have a choice, then we take any color of two possible ones). For each $i$ assign color $d_i$ to pendant vertex $u_i$. The hat colors for neighbours of sage $v$ are already given, so, his answer by the strategy is known. Assign to vertex $v$ first or second color that does not coincide with this answer. The sages have lost.

**1.9.** This statement is obtained by applying the constructor from problem 3.4, where graph consisting of one vertex with hatness 1 is taken as $G_2$! Indeed, the sages win on the path drawn in fig. 2, and the graph, described in the problem, is obtained by the addition of one-vertex graph $G_2$ to this path, and in the place of gluing the hatnesses increase by 1.

**1.10.** This statement is obtained by applying the constructor from problem 3.5, Indeed, the sages win on the path drawn in fig. 2, and the graph, described in the problem, is obtained by addition new edge with hatnesses of vertices 2 and 3 to this path and appropriate change of vertex hatnesses, to which they are fastened.

**1.11.** Answer: no.

Let $\langle G, a\rangle$ be simple winning graph, $A \in V(G)$.

Give hat of color 0 to sage $A$. Then for the other sages strategy on a losing graph $G \setminus A$ has been fixed. So there is a losing hat placement for the sages on the graph $G \setminus A$, give it to them. In the obtained hats placement on the whole graph $G$ only sage $A$ can guess correctly.

**1.12.** Present the judge's strategy. First consider 51 hats placements on the part $A$, consisting of 50 sages, where all sages of $A$ will obtain hats of the same color, and this color is one of the first 51 colors.

Will put on each of the sages of the second part $B$ the hat, that he will say by his strategy for no one of 51 placements. Fix constructed on the part $B$ placement and look what the sages of $A$ will say by their strategy. There are 50 sages, therefore, some color from the first 51 ones nobody says. Put hats of this color on all the sages of part $A$. As a result nobody of the sages has guessed correctly.

The presented reasoning works for all bipartite graphs, in the smaller part of which there are no more than $k-2$ vertices, where $k$ is hatness of the sages. If one of the parts contains $k-1$ vertices, then the sages win in the case, when the size of the second part is very large.

**1.13.** Answer: yes, the sages win.

Let sage $A$ plays by the strategy «If I see 2, then I say 2, otherwise I say 0», and sage $B$ by the strategy «If I see 2, then I say 2, otherwise I say 1», Sages $C$ and $D$ say 2, if they see hat of color 0 or 1 on their lonely neighbour ($A$ or $B$). In the opposite case they assume that they have hats not of color 2, and play the game from problem 1.5 with the other sages.

By this strategy the sages really win. Because either sages $A$ и $B$ have hat of color 0 or 1 and then somebody of these four wins, or they have hat of color 2 and then sages $C$ and $D$ have exactly not 2 (otherwise $A$ or $B$ have guessed correctly), and therefore $C$ and $D$ will win on the path between them.

**1.14.** Suppose that the sages win with the hint. Fix strategies for all sages except $A$, that they use in game with hint, and will show, how to give the strategy of sage $A$ in order to the sages win without hint.

Assume that if we assign hat of color $x$ to sage $A$ there exists the hats placement on the whole graph, in which sage $A$ has obtained color $x$, the neighbours of sage $A$ have obtained colors $u, v, w, \ldots$, the other sages have also obtained some colors, and nobody of the sages (excluding $A$) has guessed correctly. Then we want sage $A$ to guess his color correctly in this situation , i.e. his strategy must satisfy the requirement $f_A(u, v, w, \ldots) = x$.

These requirements obtained for different placements, do not contradict to each other. Indeed, if there exists another placement, where neighbours still have colors $u, v, w, \ldots$, and sage $A$ obtained another color $y$, then the sages could not win with hint $A^*$, since, having these two placements in mind, the judge can tell sage $A$, that the color of his hat is either $x$ or $y$, and after that realize the placement for which $A$'s guess is incorrect.

### Game on clique

**2.1.** Answer: yes.

For example, «bow» (fig. 15). By the constructor from problem 3.4 one can to construct similar examples for any $n$.

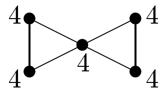



Figure 15. Bow

**2.2.** Since $i$-th sage guesses correctly on $\frac{1}{a_i}$-th part of all placements, if the sum is less than 1, there exists placement, where nobody guesses correctly.

Prove that if the sum is greater or equal to 1, then the sages win. We suggest two solutions.

Solution 1 (Hall's theorem). Fix $i$ (the number of some sage) and partition the set of all hats placements into subsets of $a_i$ elements. In each hats placement we delete color $c_i$ and for the remaining set $c = (c_1, \ldots, c_{i-1}, \widehat{c_i}, c_{i+1}, \ldots, c_n)$ (symbol "hat" means that this color is omitted) let

$$A^i_c = \{(c_1, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n) : x \in \mathrm{Col}(A_i)\}.$$

Keeping in mind application of Hall's marriage theorem, call the sets $A^i_c$ «girls», and placements themselves call «boys». Will say that boy $s$ and girl $A^i_c$ know each other, if the hats placement $s$ is an element of the set $A^i_c$. Each boy knows $n$ girls, and for each $i$ each boy knows exactly one girl of type $A^i_c$. Each girl $A^i_c$ knows exactly $a_i$ boys.

Prove that there exists matching sending each boy to a girl. For this it is enough to check the theorem condition that each $m$ boys know together at least $m$ girls. Consider an arbitrary set of $m$ boys. Since for each $i$ girl $A^i_c$ knows exactly $a_i$ boys, then for each $i$ $m$ boys know in total at least $m/a_i$ girls of kind $A^i_c$. Summing over $i$, obtain that the total number of girls which are familiar with these $m$ boys is not less than $\frac{m}{a_1} + \frac{m}{a_2} + \ldots + \frac{m}{a_n} \geqslant m$. The condition of Hall's theorem holds.

Thus, there exists a matching that put into correspondence to every hat placement a set of kind $A^i_c$. Note that, when equality $\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n} = 1$ holds, this matching in fact selects one element in each set $A^i_c$. Otherwise, if inequality $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} > 1$ holds, then «there will be lonely girls», i. e. no elements are selected in some sets $A^i_c$.

The constructed matching allows to define sages' strategy. Let $j$-th sage acts by the rule: looking at hats of the other sages, i. e. the set of colors $c = (c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n)$, he reconstructs the set $A^j_c$, which is really consists of all possible ways to supplement set $c$ to the hats placement on the whole graph. Observe that the current hats placement is one of the elements of this set. The sage must say the color that is marked in set $A^j_c$ by our matching (if there is no marked element, he says color arbitrarily).

Since each hats placement is mapped by our matching to the selected element of one of sets $A^i_c$, for this hats placement $i$-th sage will guess correctly his own color.

Solution 2 (explicit strategy). Let $N = \mathrm{LCM}(a_1, a_2, \ldots, a_n)$ and for $k$ from 1 to $n$ let $d_k = N/a_k$. We identify the set of possible hat colors of $k$-th sage and the set of remainders $d_k$, $2d_k$, $\ldots$, $a_kd_k$ modulo $N$.

Let hats be given to the sages: $k$-th sage obtains hat of color $x_kd_k$, where $x_k \in \{1, 2, \ldots, a_k\}$. Let $S = x_1d_1 + x_2d_2 + \ldots + x_kd_k \pmod N$. Each sage looking around can write all the summands of this sum except his own one. Making assumption about the value of the sum, he can calculate the color of his own hat. Let the first sage check hypothesis $S \in \{1, 2, \ldots, d_1\}$; the second sage check hypothesis $S \in \{d_1 + 1, d_1 + 2, \ldots, d_1 + d_2\}$ and so on, the $n$-th sage check hypothesis $S \in \{d_1 + d_2 + \ldots + d_{n-1} + 1, \ldots, d_1 + d_2 + \ldots + d_{n-1} + d_n\}$. The hypothesis of $k$-th sage concerns $d_k$ consecutive remainders, exactly one among them is divisible by $d_k$. This remainder defines the color of hat that the $k$-th sage should say.

**2.3.** Answer: exact strategies exist only on cliques and only when the equality

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1 \qquad (1)$$

holds.

If the graph contains two non adjacent vertices $A$ and $B$, then put arbitrary hats to all sages except $A$ and $B$. Now the answers of $A$ and $B$ are defined by the strategy. Give them hats for which their guesses are correctly. On the constructed hats placement $A$, $B$ and, possibly, somebody else guesses correctly. Therefore, the strategy is not exact. Hence the graph is a clique. It follows from the proof of problem 2.2 that for cliques the existence of an exact strategy is equivalent to equality (1).

**2.4.** Answer: no, the sages lose.

A pair of sages $A$, $B$ can be interpreted as a chess player with $n \times n$ board. It is easy to see that the fraction of total number of placements, for which he wins, equals $\frac{2}{n} - \frac{1}{n^2}$. As for the other sages, each of them wins on $\frac{1}{n}$-th part of all placements. Therefore, the total fraction of all placements, on which somebody wins, is not less than 1.

**2.5.** Answer: yes, the sages win.

We interpret hat colors of sages $A$ and $B$ as residues modulo 6, a color $C$ as a residue modulo 2, a color $D$ as a residue modulo 3. Denote hat colors of sages $A$ and $B$ by $a$ and $b$. Let sage $C$ say color $c = (a+b) \bmod 2$, sage $D$ say color $d = (a+b) \bmod 3$. If sages $C$ and $D$ have not guessed correctly, the equality $a + b = c + 1 \bmod 2$ holds and also one of the following equalities holds: $a + b = d + 1 \bmod 3$

or $a+b=d+2 \bmod 3$. Then let $A$ compute his own color assuming that $a+b=c+1 \bmod 2$ and $a+b=d+1 \bmod 3$; and $B$ compute assuming that $a+b=c+1 \bmod 2$ and $a+b=d+2 \bmod 3$.

We offer to the reader obtain the same result as an exercise using the constructor from problem 3.3.

**2.6.** Let $X$ be the set of hats placements for the first $n-2$ sages, i.e., in other words, that is the collection of sets of $n-2$ colors, where the first color is a possible hat color of sage $A_1$, the second color is a possible hat color of sage $A_2$ and so on, the $(n-2)$-th color is a possible hat color of sage $A_{n-2}$. Let $\alpha = a_1 a_2 \dots a_{n-2}$, then $|X| = \alpha$. Denote by $L_i$ $(i = 1, 2, \dots, a_{n-1})$ the subsets of $X$, such that if sage $A_{n-1}$ sees on his neighbours the set of colors from $L_i$, then he says color $i$. Similarly define sets $R_j$ $(j = 1, 2, \dots, a_n)$ for sage $A_n$. Let $L_k$ be the set $L_i$ of minimum cardinality, $|L_k| = M \leqslant \frac{\alpha}{a_{n-1}}$. Now consider sets $R_j \setminus L_k$ $(j = 1, 2, \dots, a_n)$. In these sets there are $\alpha - M$ elements in total, so if $R_m \setminus L_k$ is the set of minimum cardinality, then $|R_m \setminus L_k| \leqslant \frac{\alpha - M}{a_n}$. Therefore,

$$|L_k \cup R_m| \leqslant M + \frac{\alpha - M}{a_n} = \frac{\alpha}{a_n} + M\left(1 - \frac{1}{a_n}\right) \leqslant \frac{\alpha}{a_n} + \frac{\alpha}{a_{n-1}}\left(1 - \frac{1}{a_n}\right) = \tag{1}$$

$$= \alpha\left(\frac{1}{a_{n-1}} + \frac{1}{a_n} - \frac{1}{a_{n-1}a_n}\right) = \alpha\left(1 - \sum_{i=1}^{n-2} \frac{1}{a_i}\right) = \alpha - \frac{\alpha}{a_1} - \dots - \frac{\alpha}{a_{n-2}}.$$

Thus, when sage $A_{n-1}$ has hat of color $k$, and sage $A_n$ has hat of color $m$, somebody of sages $A_1$, $A_2$, $\dots A_{n-2}$ will win, and the number of placements, for which this event happens, equals the fraction $\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-2}}$ of total number of placements. But as we know, the indicated fraction bounds from above the number of placements, for which sages $A_1$, $A_2$, $\dots A_{n-2}$ win. Therefore, both inequalities (1) must be equalities. Then $|L_k| = \frac{\alpha}{a_{n-1}}$ (and in general $|L_i| = \frac{\alpha}{a_{n-1}}$ for all $i$), and $|R_m \setminus L_k| = \frac{\alpha}{a_n} - \frac{\alpha}{a_{n-1}a_n}$. Analogously $|R_j| = \frac{\alpha}{a_n}$. Therefore $|R_m \cap L_k| = \frac{\alpha}{a_{n-1}a_n}$, and $\alpha$ is divisible by $a_{n-1}a_n$.

**2.7.** Answer: yes, the sages win.

Let $G_1$ and $G_2$ be 4-cliques in the bow, $A$ be their common vertex, colors of the sages be the remainders modulo 5.

Let the sages on $G_1 \setminus A$ compute the sum of remainders-colors on $G_1$ and play according hypotheses that the sum has reminder, say, for 2, 3 and 4 modulo 5. The sages on $G_2 \setminus A$ play similarly.

What sage $A$ have do? He should play checking two hypotheses on $G_1$ and two hypotheses on $G_2$ at the same moment. The only that rescues him is that both hypotheses are about his unhappy hat! The strategy will work if we could convert colors of $A$'s hat to remainders modulo 5 in different ways:

| Color | Red | Blue | White | Yellow | Black |
|---|---|---|---|---|---|
| His code in $G_1$ | 0 | 1 | 2 | 3 | 4 |
| His code in $G_2$ | 0 | 2 | 4 | 1 | 3 |

The presented coding has the property that any pair consisting of successive remainders in row $G_1$, intersects with any pair of successive remainders in row $G_2$ in no more than one element. Sage $A$ computes two successive remainders corresponding to the hypotheses “the sum of all remainders on $G_1$ equals 0 or 1» and «the sum of all remainders on $G_2$ equals 0 or 1”, and says color of the intersection of these pairs, if it exists.

**2.8.** Answer: yes, the sages win here too.

Consider residues modulo $740 = 4 \cdot 5 \cdot 37$. Let hat colors of sage with hatness $k$ be the residues divisible by $\frac{740}{k}$. Denote by $S_1$ and $S_2$ the sums of residues-colors in the left and the right 5-cliques. Let the left sage of hatness 4 suppose that $S_1$ belongs to the set $\{1, 2, \dots, 185\}$. He obtains 185 consecutive residues, exactly one of them is divisible by 185. The sage says the color, corresponding to this residue. Similarly three left sages with hatness 5 suppose that $S_1$ belongs to sets $\{186, \dots 333\}$, $\{334, \dots 481\}$, $\{482, \dots 629\}$ correspondingly. The right sages acts similarly, but working with $S_2$. If now nobody has won, then $A$ (the sage with hatness 37) understands that $S_1$ belongs to $\{630, \dots 720\}$, so he needs to choose from at most 6 consecutive colors. The same can be said about $S_2$. Let the sages of left and right cliques convert the colors of sage $A$ to the residues by different ways. By our rule the hat colors of sage $A$ are the residues modulo 740, divisible by 20, i.e. really this is the residues modulo 37. If the sages of left clique convert a color to residue $x$, then the sages of right clique convert the same color to residue $6x \pmod{37}$ (the map $x \mapsto 6x$ is 1-to-1 on the sets of residues modulo 37). As it is easy to see, any two sets of the form $\{x, x+1, \ldots, x+5\}$ and $\{6y, 6y+6, \ldots, 6y+30\}$ intersect in at most one one element. Then $A$ calls the intersection color of these sets (or calls an arbitrary color if the intersection is empty).

**Constructors**

**3.1.** Hat color of sage $v$ in the graph $G_1 \times_v G_2$ can be interpreted as a pair of colors $(c_1, c_2)$, where $c_i$ is the hat color of $v$ in graph $G_i$. Fix winning strategies for graphs $G_1$ and $G_2$. Construct strategy on $G_1 \times_v G_2$: let all sages except $v$, play by the winning strategy for the corresponding graph (the neighbours of $v$ in $G_i$ look only at the component $c_i$ of $v$'s composite color). As to sage $v$, he plays by both strategies giving two answers $c_1$ and $c_2$ independently; answer $c_i$ corresponds to his winning strategy for graph $G_i$ (for computing this answer sage $v$ looks only on his neighbours from graph $G_i$). Pair $(c_1, c_2)$ is concrete hat color of sage $v$ on the graph $G_1 \times_v G_2$, that is the answer of sage $v$ in the constructing strategy.

The constructed strategy is winning because either somebody from $G_1$ or from $G_2$ will guess correctly his own color, or $v$ will guess correctly both components of his own color.

**3.2.** Let $f_1$ and $f_2$ be winning strategies from games on graphs $G_1$ and $G_2$ correspondingly.

Let also each sage $u$ of subgraph $G_2$ of graph $G$ obtain composite color $(c_1, c_2)$, where $0 \leqslant c_1 \leqslant h_1(v) - 1$, $0 \leqslant c_2 \leqslant h_2(u) - 1$. Then all «left halves» of these sages play strategy $f_1(v)$, and «right halves» of these sages play strategy $f_2$. In particular it means that all sages from subgraph $G_2$ name colors, that have the same first component.

As for the other sages from $G$, those of them, who are not the neighbours of $v$, play by the strategy $f_1$. The sages from $G_1$, that are the neighbours of sage $v$, after the substitution discovered[found out] that instead of one neighbour of $v$ they had now $|V_2|$ neighbours (and, generally speaking, with different hats). These sages act by the following way: all they instead of one hat of $v$ see all the hats on subgraph $G_2$ and know the strategies of sages on that[this] graph. Therefore, they understand who wins in the game on subgraph $G_2$, denote this player by $v_{\text{new}}$ (if there are several winners, then they choose one of the winners as $v_{\text{new}}$, for example, the winner that is in the first place in the list, drawn up before). As a result, each former neighbour of $v$ looks only at $v_{\text{new}}$, precisely, on the first component of his color, and also plays by the strategy $f_1$.

As a result either somebody guesses correctly from subgraph $G_1 \setminus \{v\}$, or $v_{\text{new}}$ guesses correctly the left component of his own color, and the right component he guesses by his definition. Thus, somebody will guess correctly his own color.

**3.3.** Hat colors of players $B$ and $C$ can be interpreted as pairs in the form $(c, \epsilon)$, where $c$ is possible hat color in the game $\langle G, h \rangle$, $\epsilon \in \{0, 1\}$. Let sage $A$ play according hypothesis $c(A) = \epsilon_B + \epsilon_C \pmod 2$. Sages $B$ and $C$ see their neighbours in graph $G$ and know what colors $c(B)$, $c(C)$ they must name by winning strategy in game $\langle G, h \rangle$. Looking at hat of sage $A$, and also the hats of each other, sages $B$ and $C$ can compute what values $\epsilon_B$ and $\epsilon_C$ they must take in addition to $c(B)$, $c(C)$.

**3.4.** By comparison with initial graphs one new color has been added for sages $A_i$ and $B_j$, we mean that this color is red. Let «megasage» $A$ says that he has all hats red, if he sees at least one red hat on sage $B$, in the opposite case let $A$ play by usual strategy on graph $G_1$. If «megasage» $B$ sees at least one red hat on $A$, then he understands that $A$ has won, if $B$ has at least one red hat. Then $B$ must care about the placements, where he has no red hats, and simply plays his strategy on graph $G_2$. If «megasage» $B$ does not see red hats on $A$, then he understands that $A$ has won, only if $B$ does not have red hats too, then, for checking the remained placements, $B$ says that all his hats are red.

**3.5.** We show strategy for new vertices and partially for old ones, then consider several placements, on which they win «immediately», and then say that on the other ones they win with the help of strategy for graph $G$.

Describe a winning strategy. Denote by $c_x$ the hat color which sage $x$ has obtained. The color of vertex $Z$ we will consider as «composite»: $c_Z = (\epsilon, C)$, where the first «bit» $\epsilon$ can take on values 0 and 1, and the second color $C$ can take on those $h(Z)$ colors that were in graph $G$ initially.

- Sage $A$, in the case when he sees hat of color 0 or 1 on sage $B$, says what he sees, otherwise he says the first bit of color $c_Z$.

- Sage $B$, in the case when he sees hat of new color on sage $C$, says 2, otherwise he says the value of $1 - c_A$.

Consider all variants of pairs $(c_A, C_B)$ and describe the other part of strategy, also proving that it is winning. In the cases $(0,0)$ and $(1,1)$ sage $A$ immediately guesses correctly. In the cases $(0,1)$ and $(1,0)$, if we will not give hat of new color to sage $C$, then sage $B$ will guess correctly. So sage $C$, seeing on sage $B$ hat of color different from 2, can fearlessly name hat of new color and somebody of $A$, $B$, $C$ will guess correctly. It remains to consider the cases $(0,2)$ and $(1,2)$. Sage $A$ will guess correctly in them, if his color coincides with the first bit of $c_Z$, therefore $Z$ can mean that his first bit differs from color $c_A$, that is he needs to guess only the second «bit», i.e. $h(Z)$ possible values. At the same time sage $B$ in these two cases will guess correctly only in the case, when sage $C$ has new color. That is $C$, seeing hat of color 2 on sage $B$, can assume that his color is not new color. Thus, the sages need to win in the game $\langle G, h\rangle$, that we are able to do by the condition.

**3.6.** The statement immediately follows from the theorem about product (product 3.1).

**3.7.** Let $A$ be new pendant vertex, $B$ be a neighbour to it vertex of graph $G$, denote the graph with added vertex by $G_1$.

In one direction the statement is evident. If the game on $G$ is winning, then game on $G_1$ is also winning. Prove now that if the game on $G_1$ is winning, then the game on $G_1$ is also winning.

Let the sages choose a winning strategy on $G_1$. Remind that in problem 1.14 it has been proved that if during the test the judge gives to one of the sages, say $B$, hint in the form «you have hat of one of two colors $c_1$ or $c_2$», then this hint does not affect the game result. Denote this hint by $B^*$.

Prove that if the sages win in the game on $G_1$, then they can win on graph $G$ with hint $B^*$. Fix a winning strategy $f$ on graph $G_1$. Construct a winning strategy on $G$ with hint $B^*$. Let all sages from $V(G) \setminus B$ use strategy $f$. For each pair of colors $(b_1, b_2)$, $b_1 \neq b_2$, that can be given to sage $B$, find such a color $a \in \operatorname{Col} A$ that $f_A(b_1) \neq a$, $f_A(b_2) \neq a$, there exists such color, since $\widehat{A} \geqslant 3$.

Now specify the strategy of sage $B$ on graph $G$ taking into account the hint, namely, let in the case, when $B$ sees the set of colors $c$ on the heads of his neighbours on graph $G$ and obtains the hint «your hat is of color $b_1$ or $b_2$», he gives the answer $f_B(a, c)$, i. e. answers by strategy $f$, as he saw on $G_1$ color $a$ on the head of sage $A$ and set of colors $c$ on the heads of other neighbours. And it can be turned out that the color that $B$ says, does not coincide with $b_1$ and with $b_2$.

This strategy is winning because when in the game on $G_1$ we give hat of color $a$ to sage $A$ during the test and observe that he does not guess correctly (not giving some hats to $B$), then for sage $B$ colors $b_1$ and $b_2$ are remained possible, and on graph $G_1$ somebody guesses correctly. Thus, the sages win with hint $B^*$.

**3.8.** Prove the more general case, from which both statements immediately follow.

The addition of two new vertices $B$ and $C$ and edges $AB$, $BC$, $CA$ to graph $G$ with vertex $A$ does not affect whether graph is winning or not, if the hatnesses of new vertices satisfy to the condition $2(\widehat{B} + \widehat{C}) < \widehat{B} \cdot \widehat{C}$ (the hatnesses of vertices of graph $G$ have not been changed).

Let $G'$ be the obtained graph. If the sages win on graph $G$, then they win on $G'$ too. Verify that if $G'$ is winning, then $G$ is winning.

Let $f'$ be a winning strategy on $G'$. Construct $f$, the winning strategy on $G$. For the sages from $G \setminus A$ the strategy remains the same. Set the strategy of sage $A$. Let a hats placement on graph $G$ be given. Sage $A$ sees the colors of all his neighbours in graph $G$ and considers $\widehat{B} \cdot \widehat{C}$ ways to choose colors for $B$ and $C$. Further sage $A$ examines what he has to answer in all these cases according to strategy $f'$, and calls the color that is the most common in his answers.

Show that the obtained strategy wins on graph $G$. Fix a hats placement on graph $G$ and suppose that no one of the sages from $G \setminus A$ has guessed correctly. If put hats on the heads of sages $B$ and $C$ arbitrarily, then we will obtain a hats placement on $G'$, in which all the sages from $G \setminus A$ has not guessed correctly their colors by strategy $f'$. Therefore, $A$, $B$ or $C$ will guess correctly. But from available $\widehat{B} \cdot \widehat{C}$ ways of hats placement on the heads of $B$ and $C$ sage $B$ will guess correctly in $\widehat{C}$ cases, and sage $C$ in $\widehat{B}$ cases. Therefore, $A$ has to guess correctly his color in at least $\widehat{B}\cdot\widehat{C}-(\widehat{B}+\widehat{C}) > \frac{\widehat{B}\cdot\widehat{C}}{2}$ cases. Hence, sage $A$ guesses his color in the most of cases, and he will guess just this color by strategy $f$.

**3.9.** Denote by $N_i$ the set of neighbours of sage $A$ in graph $G_i$, and by $S$ the set of all possible hats placements of the sages from $N_2$. If $x$ is one of two possible hat colors of sage $A$, then the second color we will denote by $\overline{x}$.

Fix arbitrary strategy $f$ for the sages on graph $G$ and describe devil's(sometimes we call the judge so) strategy, by which he will can to outplay the sages.

Choose arbitrary hats placement $s \in S$ for the neighbours of $A$ in subgraph $G_2$. This placement is uniquely sets strategy $f^s$ of the sages in subgraph $G_1$, which is, as we know, losing on this subgraph. Choose arbitrary disproving hats placement $\varphi_s$ on graph $G_1$ for strategy $f^s$. Then sage $A$ obtains hat $\varphi_s(A)$ and has not guessed correctly, it means that the chosen hats $\varphi_s$ defines such a hats placement $t = \varphi_s\big|_{N_1}$ for the sages from $N_1$, that $f_A(t,s) = f^s_A(t) = \overline{\varphi_s(A)}$.

Let the devil construct disproving hats placements on all graph $G$ applying the following principle: if in the constructing disproving hats placement it is assumed to give hats placement $s$ to the sages from $N_2$, then hats placement $\varphi_s$ will be given to all the sages in subgraph $G_1$. In the observance of this principle, first, sage $A$ and all the other sages on subgraph $G_1$ have not certainly guessed correctly colors of their own hats, and second, the strategy of sage $A$ is now completely defined only by hats placements from $S$ (since on component $G_1$ we immediately set placement $h_s$ and do not consider any other variants[cases]).

So, the devil sees that the sages from $V(G_2)\setminus A$ apply strategy $f$, and sage $A$ really use the strategy «if I see placement $s$ on the heads of my neighbours from $N_2$, then I say color $\overline{\varphi_s(A)}$». Since graph $G_2$ is losing, then there exists a disproving hats placement $\psi$ on subgraph $G_2$ for this strategy. This hats placement allows to the devil to set correctly a disproving hats placement on all graph $G$. Indeed, hats placement $\psi$ defines the set $s = \psi\big|_{N_2}$ of hat colors for the neighbours of $A$ in subgraph $G_2$, and set $s$ defines a disproving hats placement $\varphi_s$ on subgraph $G_1$, and placements $\psi$ and $\varphi_s$ are compatible: both placements assign color $\varphi_s(A)$ to sage $A$, whereas strategy $f$ requires to call color $\overline{\varphi_s(A)}$.

**3.10.** Let the sages fix strategy $f$ on graph $G'$. Construct a losing placement for this strategy. Look at the strategy of sage $A$ for this. He says one of two colors less often, namely, no more than $h(B)-1$ times. Give the hat of this color to him. Now, in order to prevent him to guess correctly, we must give to $B$ the hat of one of the remaining $h(B)$ colors (if more colors remain, select $h(b)$ colors). That is good, we just know how to give a placement on the remained graph, where $B$ has only $h(B)$ colors. That is a disproving hats placement on the whole graph.

**Blind chess**

**4.1.**

**W1)** It is trivial. In the language of hats it is a game, where some of the sages always obtains hat of the same color, certainly, he will guess it correctly, even not looking at the others.

**W2)** In the language of hats in the corresponding 4-cycle the hatness of two neighbour sages equals two, these two sages will provide win, even not looking at the others.

**W3)** This statement is a retelling to the language of the game «Check by rook» of the statement that the sages win on 4-cycle, if they all obtain hats of three colors. For example, the strategy of the sages, described in [1], in the chess language looks by the following way. If chess player sees that the king of the fellow is located in the centre, he puts the rook to the centre. In the opposite case he puts the rook to the cell, where the arrow points, leading from the king (on the auxiliary diagram for this chess player), see fig. 16. The coordinates of cells in the figure correspond to the numbers of hat colors from [1]. So, chess player $\mathcal{L}$, seeing that the king of the fellow is located in cell $(2, 2)$, puts his rook on cell $(1, 0)$ (this case corresponds to bold arrow in left fig. 16).


Strategy of $\mathcal{L}$ Strategy of $\mathcal{R}$

Figure 16.

**W4)** Number the cells of board $L(2 \times 3)$ from left to right from top to bottom, fig. 17*a*. Let the strategy of chess player $\mathcal{R}$ is given by the table in fig. 17*b*. Here six labels have been put in the cells of board $R(3 \times 4)$. Label $r_i$ means that chess player $\mathcal{R}$, seeing that the king of the fellow is located in $i$-th cell of board $L(2 \times 3)$, puts the rook on the cell of board $R(3 \times 4)$ with label $r_i$.

The strategy of chess player $\mathcal{L}$ we also set with the help of board $3 \times 4$, see fig. 17*c*. Here there is a number from 1 to 6 in each cell of board $R(3 \times 4)$, this number denotes some cell of board $L(2 \times 3)$. When chess player $\mathcal{L}$ sees that the king is located on board $R(3 \times 4)$ in the cell with label $k$, he puts the rook on the cell with number $k$ of board $L(2 \times 3)$. To avoid misunderstandings in the notations, we use the labels of type «letter $r$ with index» for chess player $\mathcal{R}$, and the labels of type «number» for chess player $\mathcal{L}$.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

a) Labelling of board $L$

| | | | |
|---|---|---|---|
| $r_4$ | $r_5$ | | |
| $r_6$ | | | $r_3$ |
| | | $r_2$ | $r_1$ |

b) The strategy of chess player $\mathcal{R}$

| | | | |
|---|---|---|---|
| 1 | 3 | 3 | 5 |
| 2 | 1 | 4 | 5 |
| 2 | 6 | 6 | 4 |

c) The strategy of chess player $\mathcal{L}$

Figure 17. The winning strategy for game $L(2 \times 3)$, $R(3 \times 4)$

Describe how winning strategies can be set with the help of the introduced notations.

S t a t e m e n t. A strategy of chess players is winning if and only if for any three different cells $a$, $b$, $c$ of board $L(2 \times 3)$ such that cells $b$ and $c$ do not belong to the cross of cell $a$, the following property on board $R$ holds: *all the cells of board $R(3 \times 4)$, marked by number $a$, belong to the intersection of the crosses $r_b$ and $r_c$.*

For example for $a = 1$, $b = 5$, $c = 6$ the cells with label 1 on board $R(3 \times 4)$ are located in the intersection of crosses $r_5$ and $r_6$. (The intersection of crosses $r_5$ and $r_6$ is painted in fig. 17*b*.)

Proof of the statement. Let the judge put the king on the cell of board $R(3 \times 4)$ labeled by 1 (for the other labels the reasoning are similar), this case specifies to us the triple of cells $a = 1$, $b = 5$, $c = 6$. Then chess player $L$ puts the rook on cell 1 of board $L(2 \times 3)$ by his strategy. Further, let the judge put the king on 5-th or on 6-th cell of board $L(2 \times 3)$, only in these cases the king on board $L$ will not be in check. Therefore, in these situations check is provided by chess player $\mathcal{R}$. It means that the cell with label 1 must belong to the cross of cell $r_5$ and similarly the cell with label 1 must belong to the cross of cell $r_6$. The statement is proved.

It remains to note (trying all possible cases) that the statement holds for the given example.

**W5)** The strategy is specified similarly to the case W4), see fig. 18.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |

a) Labelling of board $L$

| | | |
|---|---|---|
| $r_5$ | | $r_6$ |
| | $r_1 r_2$ $r_3$ | |
| $r_8$ | $r_4$ | $r_7$ |

b) The strategy of chess player $\mathcal{R}$

| | | |
|---|---|---|
| 3 | 5 | 4 |
| 8 | 6 | 8 |
| 2 | 7 | 1 |

c) The strategy of chess player $\mathcal{L}$

Figure 18. The winning strategy for the game $L(2 \times 4)$, $R(3 \times 3)$

**W6)** In the language of hats this case means that the cycle contains path $P_3$ with hatnesses of vertices 2, $x$, 2, where $x \leqslant 4$. The sages win on such path.

**L1)** As in item W4), number the cells of board $L(2 \times 3)$ from left to right from top to bottom, fig. 19*a*. Then the strategy of chess player $\mathcal{R}$ is specified by the table, similar to the one shown in fig. 19*b*. Note that for this way to specify the strategy it is allowed that several labels $r_i$ are on the same cell. The strategy of chess player $\mathcal{L}$ can also be specified with the help of board $4 \times 4$, for example, as it is done in fig. 19*c*.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

a) Labelling of board $L$

| | | | |
|---|---|---|---|
| $r_4$ | | $r_5$ | |
| | | | $r_3$ |
| | | $r_2$ | |
| $r_6$ | $r_1$ | | |

b)

| | | | |
|---|---|---|---|
| 1 | 6 | 2 | 6 |
| 5 | 3 | 1 | 2 |
| 4 | 3 | 6 | 3 |
| 2 | 1 | 4 | 5 |

c)

Figure 19. Two ways to specify the strategies

As in item W4, the following statement holds.

S t a t e m e n t. Strategy of chess players is winning if and only if for any three different cells $a$, $b$, $c$ of board $L(2 \times 3)$, such that cells $b$ and $c$ do not belong to the cross of cell $a$, the following property on board $R$ holds: *all cells of board $R(4 \times 4)$, marked by number $a$, belong to the intersection of crosses $r_b$ and $r_c$.*

For example, for the strategy shown in fig. 19*c*, the triple of cells $a = 1$, $b = 5$, $c = 6$ does not satisfy to the *statement*: only one of the three cells with label 1 belongs to the intersection of crosses $r_5$ and $r_6$. (The intersection of crosses $r_5$ and $r_6$ is painted in fig. 19*b*.)

Prove that the chess players have not a winning strategy.

Fix a strategy of chess player $\mathcal{R}$. Using the *statement*, try to understand, where the cells with labels 1, 2, and 3 can be located on board $R(4 \times 4)$. By the *statement*, the cells with label 1 belong to the intersection of crosses $r_5$ and $r_6$, the cells with label 2 to the intersection of crosses $r_4$ and $r_6$, and the cells with label 3 to the intersection of $r_4$ and $r_5$.

Note that the union of pairwise intersection of any three crosses (possibly, coinciding) on board $R(4 \times 4)$ contains at most 8 cells. Indeed, consider the cases.

1. If the centres of the crosses belong to different verticals and horizontals, then each pairwise intersection consists of two cells, see the example in fig. 19*b*, where the intersection of crosses $r_5$ and $r_6$ is painted; there are not more than 6 cells.

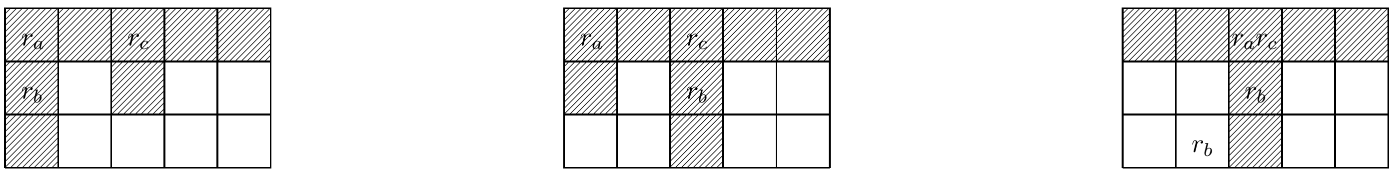
2. If the centres of any two crosses do not coincide and two centres belong to one horizontal or vertical (as for example $r_4$ and $r_5$ in fig. 19*b*, then the intersection of these two crosses contains 4 cells and adding of the third cross can give another 4 cells to the union of pairwise intersections, only if the centre of this cross belongs to the same line as one of the first two centres (as $r_4$ and $r_6$ in fig. 19*b*. In this case there are 8 cells, and 7 of them belong to one cross (in the considered example in the cross $r_4$).

3. If the centres of two crosses coincide, then the intersection of crosses contains 7 cells. For any location of the third centre the set of pairwise intersection does not increase.

Thus, for cells with labels 1, 2, 3 on board $R(4\times4)$ there are at most 8 positions, similarly for the cells with labels 4, 5, 6 there are at most 8 positions too. Since board $R(4\times4)$ contains 16 cells, we have 8 positions for labels 1, 2, 3 and 8 positions for labels 4, 5, 6. But as it was established by trying all possible cases, 8 positions can be realized only as the set «whole cross» plus one cell. It remains to note that it is impossible to cover all the board $R(4\times4)$ by two crosses and two additional cells.

**L2)** As in item L1 board $L(2\times3)$ here is the same, and the right board is also «large enough». Similarly we make sure that the union of pairwise intersections of any three crosses (possibly, coinciding) on board $R(3\times5)$ contains no more than 8 cells; the cases, in which this intersection contains 7 or 8 cells, are shown in fig. 20, this are the cases, when the centres of two crosses belong to one row or one column (including the case, when they are in one cell).

In all the cases the union of pairwise intersections of three crosses occupies one whole horizontal of the board, and in each of two other horizontals it occupies less than a half of cells. It means that the union of two such sets cannot cover the whole board.




8 cells — 8 клеток — 7 cells, $r_b$, in any place

Figure 20. The union of pairwise intersections of three crosses on board $R(3\times5)$

**L3)** This reasoning is offered by Oleg Chemokos. Fix some strategies of chess players $\mathcal{L}$ and $\mathcal{R}$ and check that such positions of kings can be found that both kings will avoid a check. Note that if a king is located in some cell of board $L(2\times4)$, then there are three «weak» cells (in another row) on the board, the cells, from which rook cannot attack this king. For winning strategy all the positions of king on board $R(3\times4)$, for which chess player $L$ puts a rook to a weak cell, must be located in one cross. And any three cells of one row can form the set of weak cells; it happens, if a king is located in the second row in the column containing the fourth cell.

The strategy of chess player $\mathcal{L}$ is a mapping $\phi\colon R(3\times4)\to L(2\times4)$ that defines what cell $\phi(K_R)$ for putting the rook will be chosen by chess player $\mathcal{L}$, if he will see the king in the cell $K_R$ of board $R(3\times4)$. Paint the cells of board $R(3\times4)$ in two colors: cell $x$, for which cell $\phi(x)$ is located in the first row of board $L(2\times4)$ we paint in white color, the others into black. Without loss of generality we may think that the number of white cells on the board is not less than the number of black cells. Consider two cases, covered all the possibilities, describing how this «not less» can be realized.

1. One of the rows of the board $R(3\times4)$ contains three white cells $r_1$, $r_2$, $r_3$ (will call this row the first) and another one row (the second) contains two white cells $r_4$ and $r_5$. Then cell $\ell$ can be chosen

in the first row of the board $L(2 \times 4)$ such that preimage $\phi^{-1}(\ell)$ is either empty set, or just one cell of the first row, say, $r_1$. In this case the other cells of the first row form a weak set, and its preimage contains cells $r_2$, $r_3$, $r_4$ and $r_5$, not belonging to one cross.

2. Each row of board $R(3 \times 4)$ contains two white cells. Then choose in the first row of the board $L(2 \times 4)$ a cell $\ell$, for which preimage $\phi^{-1}(\ell)$ consists of at most one cell (for specificity, it is located in the third row). In this case the other cells of the first row of the board $L(2 \times 4)$ form a weak set, and their four preimages, belonging to the first two rows of board $R(3 \times 4)$, form a set that cannot be covered by one cross.

**L4)** Suppose that the chess players have a winning strategy. Using the notations for describing winning strategies from item W4). Number the cells of board $L(3 \times 3)$ by numbers from 1 to 9. Then the strategy of chess player $\mathcal{R}$ is specified by a placement of nine symbols: $r_1$, $r_2$, …, $r_9$ on board $R(3 \times 4)$. And the strategy of chess player $\mathcal{L}$ is specified by writing the numbers from 1 to 9 in each cell of board $R(3 \times 4)$, which we call labels.

As in the previous item, when we put the king on cell $i$ on board $L(3 \times 3)$ there are 4 cells, from which the rook cannot attack this king. These cells and their numbers we call $i$-weak. If the strategy is winning, then it is necessary that for all $i$ the labels on board $R(3 \times 4)$, coinciding with $i$-weak numbers, are located in the cross with centre in $r_i$.

Note that symbols $r_1$, $r_5$ and $r_9$ must be located in different rows of board $R(3 \times 4)$. Indeed, it is easy to see that each cell of board $L(3 \times 3)$ is located in weak position with respect to one of the cells with numbers 1, 5 or 9. (For example, 1 and 2 are in weak position with respect to 9, 3 is in weak position with respect to 5 and so on). Therefore, each label on board $R(3 \times 4)$ is located in $r_1$-, $r_5$- or $r_9$-cross. It can be only if symbols $r_1$, $r_5$ and $r_9$ are located in different rows.

Similarly, symbols $r_i$, $r_j$ and $r_k$ are located in different rows, if cells $i$, $j$, $k$ occupy three different rows and three columns of board $L(3 \times 3)$.

Corollary. Two possible cases of placement of symbols $r_1, r_2, \dots, r_9$ on board $R(3\times4)7$ are possible: 1) either symbols $r_1$, $r_2$, $r_3$ are located in one row of board $R(3 \times 4)$, symbols $r_4$, $r_5$, $r_6$ are located in another row, and symbols $r_7$, $r_8$, $r_9$ are in the third row;

2) or symbols $r_1$, $r_4$, $r_7$ are located in one row of board $R(3 \times 4)$, symbols $r_2$, $r_5$, $r_8$ are located in another row, and symbols $r_3$, $r_6$, $r_9$ are in the third row.

The corollary is proved by gently nasty looking all the possible cases.

Prove that there are no winning strategies that have these properties. Put rooks on all cells $r_i$ of board $R(3 \times 4)$ (we put on cell as many rooks as there are symbols $r_i$ in it). By the corollary in the first row of board $R(3 \times 4)$ there is an «empty» cell, i. e. cell, containing no symbols $r_i$, but containing some label $a$. Let for the specificity it be located in the fourth column (fig. 21). By the statement, 4 rook's attacks are directed to this cell, and two of these four rooks are located in one row, and another two are in another row. It means that two rooks are certainly located in one of the cells of fourth column. Let for the specificity this cell be located in the second row. Now we know that in the second row 3 rooks have been put in total, and two of them are located in one cell.
Therefore, there are two «empty» cells in the second row. We choose one of them, above which in the first row no more than one rook is located. Let this cell be located in the first column for the specificity and contains label $b$. 4 rook's attacks from two pairs of rooks, located in two rows, are directed to the chosen cell. One pair of rooks is located, evidently, in the second row, and another pair is located in the third row (there is no more than one rook in the first row above cell $b$). Now we see that one of

| | | | |
|---|---|---|---|
| | | | $a$ |
| $b$ | | | $r_1r_2$ |
| $r_7r_8$ | × | × | |

Figure 21. The strategy for the case L3

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 |

a) Labelling of board $L$

| | | |
|---|---|---|
| 1 | 2 | |
| | | |
| 3 | 4 | |

b) The strategy of chess player $\mathcal{L}$

Figure 22. We find a strategy for the game $L(2 \times 5)$, $R(3 \times 3)$

the cells in the third row, in the second or in the third column, cannot gather 4 rook's attacks from two different rows. It is contradiction.

**L5)** Suppose that the chess players have a winning strategy. Use the notations for describing winning strategies from item W4). Number the cells of board $L(2\times 2)$ by numbers from 1 to 4. Then the strategy of chess player $\mathcal{R}$ is specified by placement of four symbols $r_1$, $r_2$, $r_3$, $r_4$ on the board $R(5\times 5)$. On board $R(5\times 5)$ at least one cell can be found, not belonging to any of four crosses, defined by these symbols; call this cell $Q$. The strategy of chess player $\mathcal{L}$ is specified by writing in each cell of board $R(5\times 5)$ the numbers from 1 to 4. Consider the number, written in cell $Q$, without loss of generality, it is 1. Consider the number on board $L(2\times 2)$, located on the same diagonal as 1, without loss of generality, it is 4. Let the judge put the kings: on cell $Q$ on board $R(5\times 5)$ and on cell 4 on board $L(2\times 2)$. Then player $\mathcal{L}$ puts the rook on cell 1 of board $L(2\times 2)$, and player $\mathcal{R}$ puts the rook on cell $r_4$ of board $R(5\times 5)$. None of the rooks has a king in check. The chess players lost.

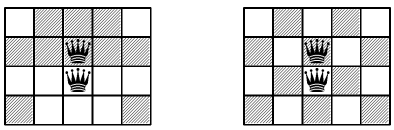
**L6)** Label board $L$, as in fig. 22 a). As in W4), the strategy of chess player $\mathcal{L}$ is specified by writing in each cell of board $R(3\times 3)$ the numbers from 1 to 10, the numbers of cells on board $L(2\times 5)$. Since there are only two horizontals in board $L(2\times 5)$, there exist two rows of board $R(3\times 3)$, in each of them the numbers of two cells are written, such that all these four cells (possibly, there are coinciding among them) belong to one horizontal of board $L(2\times 5)$. Let $j$ be the number of the cell from the second horizontal, that is $i$-weak with respect to all these cells.

For example, let labels 1, 2, 3, 4 are located on board $R(3\times 3)$, as in fig. 22 b). Then the number 10 is 1-, 2-, 3- and 4-weak simultaneously. It means that the rook on cell $r_{10}$ of board $R(3\times 3)$ attacks the cells with labels 1, 2, 3 and 4. It is impossible: it must be located in the upper row of board $R(3\times 3)$ to attack labels 1 and 2, and in the bottom row to attack 3 and 4.

By the same reason the general case is also impossible: cell $r_j$ must be located in two rows of $R(3\times 3)$ simultaneously.

**4.2.** Paint the cells of both boards as shown in fig. 23, a). Let both chess players put their queens only on the cells that occupied by the queens, and let the first chess player act according the assumption «Kings are located in cells of the same color», and the second from the assumption «The kings are located in cells of different colors».

We can use the usual chess coloring instead of «exotic» coloring as above. Indeed, the queen, located in cell $c2$, holds under attack all the cells of the same color in chessboard coloring! And the same for $c3$, fig. 23, b).



a) b)

Figure 23. «Check by queen» on boards $4\times 5$

**4.3.** Statement of the problem has been found with help of computer. The participants of our conference suggest beautiful logical strategies.

S o l u t i o n 1 (Kononenko Nukolay). Specify strategy of the chess players. Label board $R(5\times 5)$ as shown in fig. 24 a). Seeing the king on the cell with label $j$, chess player $\mathcal{L}$ puts the rook on the cell of board $L(4\times 4)$, labelled by number $j$, fig. 24 b). Therefore, chess player $\mathcal{L}$ uses only four positions for his queen. For each cell of board $L(4\times 4)$ in fig. 24 c) it is shown, from which positions the queen of chess player $\mathcal{L}$ does not attack this cell. For example, the numbers 1 and 2 in the lower left corner mean that the lower left corner cell of board $L(4\times 4)$ is not under attack by the queen located at 1-th and in 2-th positions, shown in fig. 24 b), and "–" means that the cell is under attack from all positions.

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 3 | 1 | 3 |
| 1 | 3 | 3 | 3 | 4 |
| 3 | 3 | 3 | 3 | 3 |
| 2 | 3 | 3 | 3 | 4 |
| 3 | 2 | 3 | 2 | 3 |
| $a$ | $b$ | $c$ | $d$ | $e$ |

a) The strategy of chess player $\mathcal{L}$

| | | | |
|---|---|---|---|
| | | | |
| | 1 | 3 | |
| | | 2 | |
| 4 | | | |

b) On which cell we put the queen

| | | | |
|---|---|---|---|
| 3 | 2,4 | 4 | 1,2 |
| 2 | 4 | – | 4 |
| 3 | – | 4 | 1,4 |
| 1,2 | 3 | 1 | 3 |

c) Instruction for $\mathcal{R}$

Figure 24.

Seeing the king on board $L(4 \times 4)$, chess player $\mathcal{R}$ with help of fig. 24 c) immediately understands, from which «unfavourable» positions the queen of his fellow cannot put the king in check. Therefore he must to locate his queen on board $R(5 \times 5)$ so that it attacks all the cells, sending the queen of chess player $\mathcal{L}$ to a unfavourable position.

For unfavourable positions 1, 2 it is possible to put the queen on cell $b3$, for 1, 4 on cell $c4$, for 2, 4 on cell $c2$, for 3 on cell $c3$.

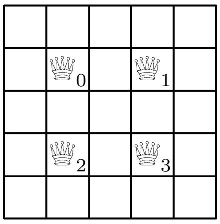
Solution 2 (Presnova Ekaterina, Raceeva Olga).

Label board $L(4 \times 4)$ as shown in fig. 25 a). Observation 1: for this labelling the property holds: for any pair of labels all the cells marked by this pair of labels, can be attacked by one queen. For example, all the cells, marked by 0 or 1, can be attacked from cell $a4$; all the cells marked by 0 or 2, can be attacked from cell $c2$ and so on.

Seeing the king in the cell with label $j$, let chess player $\mathcal{R}$ put the rook on position ♕$_j$ of board $R(5 \times 5)$, fig. 25 b). Observation 2: any cell of board $R(5 \times 5)$ can be attacked from at least two different positions from this set. Then chess player $\mathcal{L}$, seeing the king on board $R(5 \times 5)$ immediately understands, from which positions (among the indicated four positions) the queen of player $\mathcal{R}$ cannot put the king in check. In view of observation 2 there are no more than 2 such «unfavourable» positions, and chess player $\mathcal{L}$ should put his queen on board $L(4 \times 4)$ so that he attacks all the cells, sending the queen of chess player $\mathcal{L}$ on unfavourable position. It is possible, in view of observation 1.

| | | | |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 1 | 0 | 2 | 2 |
| 0 | 2 | 0 | 2 |
| 1 | 3 | 2 | 0 |

a)



b)

Figure 25.

| | | | | | |
|---|---|---|---|---|---|
| | | | | | $C$ |
| | $B$ | | | | |
| | | | | | |
| $A$ | | | | | |

Figure 26.

**4.4.** Solution 1 (Kostina Ekaterina, Mirgalimova Rozalina, Hamikova Marina). We will prove the stronger fact, that in the game «Check by queen» on boards $L(4 \times 6)$, $R(7 \times 7)$ the sages lose.

Let the judge plans to put the king on one of cells $A$, $B$, $C$ of board $L(4 \times 6)$ (fig. 26). For each of these positions player $\mathcal{R}$ is ready to put the queen on board $R(7 \times 7)$. Since three queens cannot hold in check all the cells of board $7 \times 7$, there exists a cell on board $R(7 \times 7)$, which is attacked by no one of the queens. If the judge puts the king on this cell, player $\mathcal{L}$ answers putting the queen on board $L(4 \times 6)$. Since cells $A$, $B$, $C$ cannot be attacked by one queen, at least one of them will be not under attack, and the judge, finally will formulates his plans regards board $B$: he will pat the king exactly on this cell. The chess players have lost.

**4.5.** The result is obtained by computer.

**4.6.** The result is obtained by computer.

**4.7.** Answer: yes. 5 queens can be placed on $11 \times 11$ board, that attack all the cells (for example, $b4$, $d10$, $f6$, $h2$ and $j8$). Then combine idea of the solution of problem 4.2 and standard idea of the game with hats of five colors on 5-clique.

**5.1.** Let the sage fix some strategy on new graph. The strategy of sage $A$ can be given in the form of $3 \times 7$ table: the rows correspond to the hat colors of sage $B$, the columns to the hat colors of sage $A$. In cell of table number of color (0 or 1) will be written, that sage $A$ calls, when he sees on $B$ and $C$ the corresponding hat colors.

In each column of the table one of the symbols, 0 or 1, occurs two times. Mark the cells containing repeated symbol. (If symbol occurs in all three cells of column, mark any two of them.) The marked cells can be located either in the first and the second rows, or in the first and the third. Since there are 7 columns, by the pigeonhole principle there exist two rows, in which the marked cells occupy three columns. In the marked cells of one column two zeroes and two ones can be, therefore, it is possible to choose two columns of three so that in the indicated columns in the marked cells there is the same number.

Thus, we have chosen in the table two rows (for the specificity $i$-th and $j$-th) and two columns (for the specificity $k$-th and $\ell$-th), in the intersection of which there is the same number, for the specificity it is 0. Now we without problem construct a disproving hats placement on all the graph. Give hat of color 1 to sage $A$, will choose hat of $i$-th and $j$-th color to sage $B$, $k$-th or $\ell$-th to sage $C$. For this approach sage $A$ will certainly guessed incorrectly his own color, since in according to the table he will call color 0. As to assignment of concrete colors to sages $B$ and $C$, and also for the others, consider game on graph $G$: after fixation of color of sage $A$ the strategy of other sages on graph $G$ is uniquely defined, the accepted restrictions of hat colors of $B$ and $C$ allow to suppose that their hatnesses now equal 2. Since graph $G$ is losing, we will can to present a disproving hats placement on it.

**5.2.** Prove the following a few more strong statement.

There exists positive integer $N$ such as on any graph $G$, degrees of all vertices of which do not exceed 3, and hatness function is given by formula

$$\widehat{a} = \begin{cases} 3, & \text{if } \deg a = 1, \\ 41, & \text{if } \deg a = 2, \\ N, & \text{if } \deg a = 3, \end{cases}$$

the sages lose.

*Proof.* Denote $m = 80\binom{81}{41} + 1$. Show that $N = 80\binom{m}{41}\binom{81}{41} + 1$ is suitable.

Induction by the number of vertices. Basis (graph with two vertices) is evident. Induction step. Consider the vertex of the least degree, denote it by $A$.

Case 1. $\deg A = 1$. By the statement of problem 3.7 addition of vertex of degree 3 does not affect the property of graph be winning or not, and if increase hatness of one of vertices much in losing graph, then it remains losing.

Case 2. $\deg A = 2$. Denote the neighbours of $A$ by $B$ and $C$, $\deg B \leqslant \deg C$.

Lemma. Table $(2k-1) \times \left(2(\ell-1)\binom{2k-1}{k} + 1\right)$ is painted in 2 colors. Then it is possible to choose such $k$ rows and $\ell$ columns that all these cells in the intersection of these rows and columns have been painted in the same color.

*Proof.* Consider arbitrary column, there are $2k-1$ cells in it. Then by the pigeonhole principle there exist $k$ cells of one color. Mark these $k$ cells. Make this for each column. The number of ways to mark $k$ cells of $2k-1$ equals $\binom{2k-1}{k} + 1$, these cells can be of one of two colors. Therefore, by the pigeonhole principle there exist $\ell$ columns such that the marked cells are located in the same set of rows and they are of the same color. Q.E.D. $\square$

Case 2.1. $\deg B = \deg C = 2$. Show that the sages lose in graph $G$, even if $\widehat{A} = 2$. Suppose that there exists a winning strategy of the sages on $G$. The strategy of sage $A$ is a $41 \times 41$ table, in which each cell is painted in one of two colors. Оставив от нее only 5 rows, apply the lemma for $k = \ell = 3$. 3 rows and 3 columns can be chosen from the table so that their intersections are painted in one color. Then the judge will put hat of another color on $A$, and for $B$ and $C$ will choose hats of only three colors corresponding to three rows and three columns correspondingly[?]. We will obtain the winning strategy for $G \setminus A$, $\widehat{B} = \widehat{C} = 3$ that contradict the induction hypothesis.

Case 2.2. $\deg B = 2$, $\deg C = 3$. Show that here the sages lose too on graph $G$, even if $\widehat{A} = 2$. For $k = 3$, $\ell = 41$ the table from the lemma has size $5 \times 801$, and the hatnesses of sages $B$ and $C$ are much larger: they equal 41 and $N$. Applying the lemma similarly to case 2.1, we obtain contradiction again.

Case 2.3. $\deg B = \deg C = 3$. Similarly to case 2.1, use the lemma for $k = \ell = 41$. We can use it because for $k = \ell = 41$ the table has size $81 \times N$, both dimensions are less than hatnesses of sages of $N$.

Case 3. $\deg A = 3$. Let $B$, $C$ and $D$ be the neighbours of sage $A$. Then $\deg B = \deg C = \deg D = 3$. Fix some color of sage $D$. Then the strategy of sage $A$ is a $N \times N$ table. Taking 81 rows and $m$ columns, apply the lemma, find 41 rows and 41 columns, the intersections of which are painted in one color. The number of ways to choose 41 rows and 41 columns equals $\binom{m}{41}\binom{81}{41}$. Perform these actions for each of $N$ possible colors of sage $D$. Since $N = 80\binom{m}{41}\binom{81}{41} + 1$, there exists a color of sage $D$, for which the sets of rows, columns and colors of intersections coincide. Put on sage $A$ hat of the opposite color, take 41 colors for each of the sages $B$, $C$, $D$, and obtain the contradiction to the induction hypothesis.

# References

[1] Задачи Санкт-Петербургской олипмпиады школьников по математике 2016 года. М.: МЦНМО, 2017.