Tilings: substitutions and decorations

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This project is devoted to plane tilings. Usually, if we can tile a plane with some types of tiles, then we use construction which structure is periodic. Correctly, a tiling is called *periodical* if it doesn't change after shift on non-zero distance. In 1961 chinese mathematician Hao Wang have stated the following problem:

Consider unit squares. Each square's side is colored by one color of several possible ones. Consider some types of these squares. We can attach any square to another side to side if these sides have the same color. Suppose that we can use unlimited number of each type's squares. Is it true that if we can tile a plane then we can do it periodically?

Firstly Wang expected that the answer is positive, namely if there exists some tiling then there is a periodic one. But in 1966 Robert Berger (student of Wang) have constructed a set consisting of 20426 squares such that there exist non periodical tilings only. Lately Berger reduced his set to 104 squares, and in 1971 Raphael Robinson greatly simplified the construction and introduced a set of six poligons such that one can tile a plane using them only non periodically.

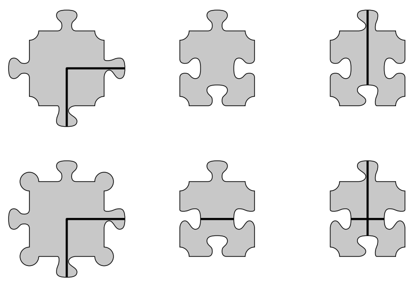


FIGURE 1. Robinson's tiles.

This problem has some philosophical background. The point is to reach global condition (non periodicity) using local sources (boundary conditions). There are many such problems. How to organize a computation network such that any local fault can not violate global processes. Or how interaction of molecules leads to crystals growing.

In this project we will study some ideas and methods to solve such problems in various settings. The main problem of first half is to construct a set of polygons such that there exist non periodical tilings only (problem B10). In fact, this problem can be reduced to constuction of local rules: boundary conditions which define when we can attach tiles to each other. This construction uses some ideas. In B and C we study some tricks that can be used to construct aperiodic tilesets.

A. One dimensional case

Consider two-side infinite tape which consists of cells (equal squares). We will assign one letter of the finite alphabet for each cell. A cell with assigned letter is called a *tile*.

If every cell contains some letter (or the tape is covered by non intersecting tiles) then we say that this is *the tiling*. A tiling is called *periodical* if it doesn't change after shift on non zero integer p. It is easy to see that periodical tiling is actually periodical repeat of some combination of p tiles. This combination is called *period* of tiling, p is length of period.

A *forbidden word* or *local rule* is a combination of several adjacent tiles. We will consider tilings which contain no forbidden words. In notations we use letters of a finite alphabet to denote tiles and we do words to denote combinations of tiles. Tilings which contain no forbidden words are called *permitted* ones. Further we assume that the number of local rules is finite.

Two tiling which transforms to each other after some shift are called *equivalent*. Below we consider such tilings as the same.

A1 Consider an alphabet $\{0, 1\}$. The word 01 is forbidden. How many non-equivalent tilings exist?

A2 Show that for two letters there exists a non periodical tiling.

A3 Construct a finite set of forbidden words in the alphabet $\{a, b, c\}$, such that the periodic tiling with period abc is the only permitted tiling.

A4 Does there exist a set of local rules such that there are exactly 100 nonequivalent permitted tilings?

A5 Let A be a finite word. Prove that there exists a of forbidden words such that the tiling with period A is the only permitted one.

A6 Let A be an aperiodic tiling. Prove that there is no a finite set of forbidden words such that A is permitted and all the permitted tilings are equivalent to A.

A7 Consider an alphabet with size n. Suppose that there are k forbidden words, and each of them has length 2. Also, there are no permitted tilings. Find the minimal possible k.

B. Dimension two: local rules and substitutions

Conclusion from one dimensional case. In one dimensional case forbidden word is some kind of local condition. Using these conditions we want to obtain some global property. But in one dimensional case finite number of local rules does not allow us to force non periodical tilings.

Let us proceed to two dimensional case. We will start with basic definitions and notation.

Consider an infinite square grid and a finite alphabet L (*alphabet of tiles*). We will write letters of L into cells of an infinite square grid, one letter to one cell. This alignment is called a *tiling*.

Similarly to one dimensional case, a tiling is called *periodic*, if it doesn't change after translation by a non zero vector (a, b), and is called *aperiodic* in the opposite case.

Equivalence of tilings. Definition. Consider two tilings S_1 and S_2. Suppose that we can translate S_1 by integer vector and obtain S_2. In this case we say that S_1 and S_2 are *equivalent. Remark.* Different tilings of finite regions are never equivalent.

Local rules. Definition. Let $n > 1$ be an integer. Consider a set of all squares $n \times n$, compiled from letter-tiles. Formally, this set is L^{n^2} (set of all $(n^2)-$tuples of letters from L). Let R be a subset of this set. So, R is a set of several squares 2×2 compiled by L letters. Let us call these squares as *forbidden* ones. A tiling is called *permitted by R* if there are no forbidden squares in the tiling. The squares from R are also called *local rules*, the number n is the *size* of the local rules.

Using local rules we can force some properties of permitted tilings. Let's do some practice.

B1 Suppose that the alphabet contains only two letters (black and white squares). Construct local rules such that the chessboard tiling is the only permitted tiling.

B2 Suppose that the alphabet contains two letters and eight squares 2×2 are forbidden (look at the picture). For given m and n, how many permitted tilings of a $m \times n$ rectangle exist?

1 1	1 1	1 0	0 1	1 0	0 1	0 0	0 0
0 1	1 0	1 1	1 1	0 0	0 0	0 1	1 0

B3 Suppose that the alphabet contains two letters. The set of local rules is the same as in the previous problem, and also the square with 1 on main diagonal (see picture below) is forbidden. Classify all different permitted tilings of the infinite plane.

1 1	1 1	1 0	0 1	1 0	0 1	0 0	0 0	1 0
0 1	1 0	1 1	1 1	0 0	0 0	0 1	1 0	0 1

B4 Suppose that alphabet contains four letters and a 2×2 square is permitted if and only if it consists of four different letters. How many permitted tilings do exist for the rectangle $m \times n$?

B5 Consider a binary alphabet. Suppose that there are k forbidden squares 2×2 and suppose that there are no permitted tilings. Find the minimal possible k.

B6 Consider some set of local rules in the alphabet $\{0, 1\}$. Suppose that the tiling on picture below is permitted (other cells except these nine contain 0). Prove that there exists infinite number of nonequivalent permitted tilings.

0	1	0
1	0	1
1	1	1

This problem show that sometimes[1] it is not possible to determine one precise tiling by local rules. But if we cannot force one tiling, maybe we can determine some set of similar tilings? We say that some set of tilings is *defined by local rules* if tilings from this set are the only permitted tilings according with these local rules.

B7 A *frame* is a figure consisting of $2(m + n) - 4$ boundary cells of a rectangle $m \times n$ for some $m, n > 1$. We call a $\{0; 1\}$-tiling *pretty*, if union of some (maybe infinite) set of frames is filled with units, and zeroes are placed in the remaining cells. Suppose we have a finite set of local rules such that all pretty tilings are permitted. Prove that there are infinitely many non-equivalent permitted not pretty tilings.

B8 Suppose we have a set of local rules, and suppose that for any positive r we can tile an area including circle of radius r such that there are no forbidden squares. Prove that we can tile the whole plane with the same condition.

[1]Further we will prove that for aperiodic tilings it is *never* possible.

Using local rules we can obtain really complicated tilings. For example, we can force all the permitted tilings to be aperiodic. It is hard to solve this problem now and we recommend to return to it in C-part and use some additional methods.

B9* Construct a set of local rules such that all permitted tilings are non periodic.

B10* Construct a finite set of polygons such that all tilings are non-periodic. All polygons can be rotated and reflected; they should be arranged without holes or overlaps.

Definition. Suppose that for any letter in the alphabet L corresponds a some square $k \times k$ compiled by letters of this alphabet. This correspondence is called a *substitution*.

For a given substitution σ and a tiling A of some (finite or infinite) region we can construct a new tiling $\sigma(A)$ by simultaneous replacement of tile-letters by corresponding squares $k \times k$. Starting with one tile and iterating this procedure, we obtain tilings of bigger and bigger squares.

Definition. Consider a $k \times k$ substitution σ. Suppose that a tiling S can be uniquely divided by horizontal and vertical lines to squares $k \times k$ such that each square is image of some letter (any square is $\sigma(a)$ for some letter a). In this case we can write a letter instead of each $k \times k$ square (a instead of $\sigma(a)$). Thus we obtain a new tiling $\sigma^{-1}(S)$. If $\sigma^{-1}(S)$ is defined in a unique way, we say that for tiling S we can find an inverse image $\sigma^{-1}(S)$. A tiling is called *infinitely decodable by* σ if one can find inverse image any number of times.

$$\boxed{1} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 0 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad \boxed{0} \longrightarrow \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

B11 Consider the substitution above. An infinitely decodable tiling for this substitution is called a *Sierpinski carpet.*

a) Prove that there are infinitely many nonequivalent Sierpinski carpets.

b) Prove that all Sierpinski carpets besides one of them are aperiodic tilings.

c) Is the set of all Sierpinski carpets defined by local rules?

If one wants to define infinitely decodable tilings with local rules, it is natural to require that a substitution map permitted tilings only to permitted tilings.

Definition. We say that a substitution σ *agrees with local rules* R of size k if the following conditions hold:

1) the square $\sigma(a)$ is permitted for every letter a;

2) if X is permitted square then the $2n \times 2n$ square $\sigma(X)$ contains only permitted $k \times k$ squares.

B12 Consider some local rules R. Suppose that $k = 2$ (all forbidden squares are 2×2). Also, suppose that a $N \times N$ square A is permitted (it contains no forbidden 2×2 squares). Let a substitution σ agree with local rules. Prove that the square $\sigma(A)$ is also permitted.

B13 Suppose that substitution σ agrees with the local rules of size 2. Prove that there exists a permitted tiling.

B14 Consider a substitution σ. Does there exist a tiling S such that $\sigma(S)$ is equivalent to S? Find sufficient conditions on σ for existence of such a tiling.

C. Examples of non periodicity

Definition. A tileset is *aperiodic*, if it admits at least one tiling of the plane but doesn't admit a periodic one.

Conclusions from parts A and B. We want to find a set of tilings such that it is defined by local rules and contains only aperiodical tilings. There is no such an example in one-dimensional case (why?). The simplest two-dimensional examples we know are infinitely decodable tilings for some substitutions.

Not any substitution is suitable.

C1 a) Find a substitution such that all infinitely decodable tilings are aperiodical, but the set of them is not defined by local rules.

b) Find a substitution such that all infinitely decodable tilings are periodical.

It is not easy to find a suitable substitution "from scratch", so we'll "improve" not suitable ones.

Transition to another alphabet. Decorations. Consider a tile alphabet $a, b, c \dots$. We can make finite number of duplicates (shades) for each letter and consider the extended alphabet of tiles $a_1, \dots a_k, b_1, \dots b_k, c_1, \dots c_k, \dots$ Now we can set up local rules for this extended alphabet. After that we can take permitted tilings and ignore the shades. This approach is called *setting up decorations.*

Definition. Let A_1 and A_2 be two tile alphabets and let σ_1 and σ_2 be two substitutions in alphabets A_1 and A_2 respectively. We call the substitution σ_2 *a decoration* for σ_1 if the following conditions are satisfied:

1) there exists a mapping f from A_2 to A_1;

2) if $\sigma_2(a) = M$ is a square made of letters from A_2, then $\sigma_1(f(a)) = f(M)$.

We consider f as "forgetting the shade", and $f(M)$ is a square obtained from M by applying f to each letter.

Our main goal is to prove that a given substitution can be decorated in such a way that the set of infinitely decodable tilings is defined by local rules.

Firstly we deal with some useful special cases. Consider a 2×2 substitution σ. If a is a letter, then $\sigma(a)$ is a 2×2 square which consists of four letters. We assume that different letters correspond to different 2×2 squares.

Images separating. Definition. Let us define σ_{UL} mapping. It maps letter a to the upper left corner of the square $\sigma(a)$. Similarly we can define mappings σ_{DL}, σ_{UR}, σ_{DR}. We say that the substitution *separate the images* (or has property of separating images) if every letter occurs exactly at one image of these four mappings.

C2 Construct a substitution with property of separating images.

The property of separating images helps us to construct local rules. Firstly we try to archive the goal for particular case – some fixed substitution. After this, we will study substitutions with separating images. And finally we will proceed to the common case.

The idea of construction. Firstly try to color sides of squares in different colors and formulate local rules in terms of these colors combinations. The main goal is to obtain the property that any permitted tiling can be decoded that is to proceed to next level of hierarchy. A substitution make this transition.

C3 Find some substitution σ and local rules such that any permitted tiling S can be decoded and tiling $\sigma^{-1}(S)$ is also permitted. Prove that any permitted tiling is aperiodical.

C4 Consider a 2×2 substitution σ with property of separating images. Suppose that there exist local rules R such that for any permitted tiling S there exists inverse image $\sigma^{-1}(S)$ which is also permitted by the same local rules R. Prove that any permitted tiling is aperiodic.

C5 Consider a 2×2 substitution with property of separating images. Prove that there is a decoration such that the set of infinitely decodable tilings is defined by local rules.

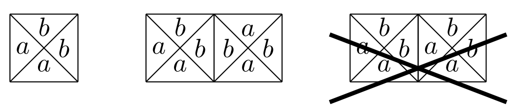
C6 Consider a 3×3 substitution with property of separating images. Prove that there is a decoration such that the set of infinitely decodable tilings is defined by local rules.

C7 Consider a 2×2 substitution (maybe there is no separating images property). Prove that there is a decoration such that the set of infinitely decodable tilings is defined by local rules.

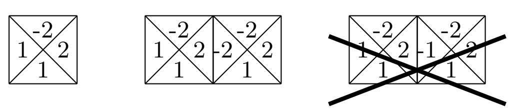
D. Other formalisms and transitions of non-periodicity to other languages

In addition to local rules and forbidden squares, there are other formalisms to specify tilings. Sometimes the interaction of different formalisms is useful.

Wang tiles formalism. There are finitely many colors, a *Wang tile* is a unit square with a color on each side. There is a finite set of Wang tiles, colors of common sides should match in tilings of the plane.



Complementary colors formalism. There is a finite set of non-zero integers (we call them colors). A tile is a unit square with an integer on each side. There is a finite set of tiles, numbers on common sides should be opposite.



D1 Prove that Wang's formalisms and complementary colors formalism are equivalent, i.e. if there is an aperiodic tileset in one formalism, then there is a corresponding aperiodic tileset in the other.

Sometimes a tileset along with any tile contains the rotated on 90 degree tile. In this case we say that we can rotate tiles.

Consider a tile A with integers (a_1, a_2, a_3, a_4) written on its sides *clockwise.* The tile with $(-a_3, -a_2, -a_1, -a_4)$ integers written on the same sides is called *flipped* tile (with respect to A). If a tileset contains the flipped tile for every tile then we say that we can flip tiles.

D2 We call a polygon *square-composed*, if it is connected polymino composed of unit squares without holes inside. Construct an aperiodic set of square-composed polygons such that with every polygon Φ the tileset contains all 8 rotations and reflections of Φ.

D3 Suppose that a tileset in complementary colors formalism with any tiles contains all its rotated and flipped tiles. a) Can such a tileset be aperiodic?

b) Add extra rule to the formalism: a tile can not be a side-neighbor of its flipped tile. Prove that such a tileset can be aperiodic.