Tilings: substitutions and decorations. Solutions.

A. One dimensional case

A1 *Answer: 3.*

It is clear that two tilings with all identical letters (ones or zeros) are permitted. If there are both 1 and 0, then there must be adjacent 1 and 0, and 1 is on the left. It's clear that only 1's can be to the left of 1, and only 0's can be to the right of 0.

1	1	1	1	1	1	1	1	1	1	1	1

0	0	0	0	0	0	0	0	0	0	0	0

1	1	1	1	1	1	0	0	0	0	0	0

A2 There are a lot of examples. For instance, one 0, and the other cells are filled with 1-s. Then for any nonzero shift this zero will map to a different one.

1	1	1	1	1	0	1	1	1	1	1	1

A3 See for yourself that if you forbid the words aa, ac, ba, bb, cb, and cc, then any letter uniquely forces both of its neighbours.

A4 *Answer: yes.* For example, we can forbid all tilings except 100 periodic sequences with periods $01, 001, \ldots, 0^{100}1$. We forbid $11, 0^{101}$, and $100 \cdot 99$ words of form 10^a10^b1 for all $a \neq b$ from 1 to 100.
The first two rules force that the number of zeroes between two neighbouring 1's is from 1 to 100. The rest of them guarantee that any two neighbouring blocks of zeroes are equal.

A5 Recall that a cyclic shift of a word $a_1a_2\ldots a_n$ is any one of the words $a_{i+1}a_{i+2}\ldots a_na_1a_2\ldots a_i$ for $0 \leqslant i < n$. Let $|A| = n$, and let k be the size of the alphabet. There are k^n words of length n; we forbid all of them that are not cyclic shifts of A. In all cyclic shifts of A, the number of letters of each type is the same, so in any permitted tiling the tiles at the distance n must coincide, that is, the permitted tiling is periodic with period length n. If some cyclic shift of A is a period, then A is also a period.

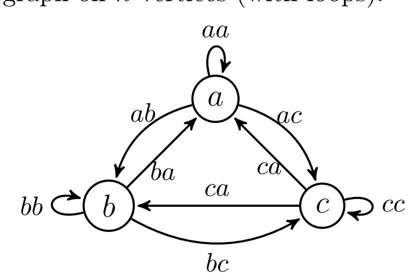
A6 We show that for any finite set of local rules that admit at least one permitted tiling of the line, there exists a permitted periodic tiling. Let N be the length of the largest of the forbidden words. Consider a permitted tiling, and by pigeonhole principle, find in it two disjoint occurrences of the same block of length N; we denote it U, and the part between the two occurrences is denoted by W.

$\cdots$	U	W	U	$\cdots$

Then the tiling with period UW is permitted. Indeed, all the rectangles of length no greater than N occur in the original tiling (and therefore do not contain forbidden ones), and we do not have local rules longer than N.

$\cdots$	W	U	W	U	W	$\cdots$

A7 *Answer:* $n(n+1)/2$.
example: Let the alphabet consist of numbers from 1 to n and the words ab are forbidden for $a \geqslant b$. Then it will not be possible to tile a block that is longer than n, because when moving from left to right, each letter should be larger than the previous one. *Estimate:* Forbidden words of length 2 are constraints of adjacent pairs of letters. Draw a complete directed graph on n vertices (with loops):



Each forbidden word of length 2 removes one edge in this graph. It is easy to understand that if there is at least one cycle left, then an infinite path along this cycle is a a permitted periodic tiling. Now we show by induction on n that if there are no oriented cycles in the oriented graph, then the edges are not more than $n(n-1)/2$. Step from n to $n+1$: if there is at least one edge from each vertex, then there is a cycle. So there is at least one vertex without outgoing edge; removing it removes at most n (incoming) edges, and leaves us with a graph with $n-1$ vertices, to which we can apply the induction hypothesis, and get $n + n(n-1)/2$ edges, which is the wanted value. And the base $n = 1$ is obvious.

B1 It is enough to forbid all the 2×2-squares except for the two chess-colored ones. Then in the permitted tiling, white cells will border only with black, and black – only with white. Conversely, the chessboard tiling is permitted.

B2 *Answer:* 2^{m+n-1}. Note that the allowed 2×2-squares are those with an even number of 1-s. Hence, each cell is uniquely determined by its three (left, top, left-top) neighbours. Therefore, for any filling of the left column and of the top row of the rectangle, the remaining cells can be filled in a unique way. Moreover, all possible binary (left, top, left-top) triples are extendable, so that every filling of the left column and of the top row is extendable as a permitted rectangle.

a_2	a_3
a_1	?

B3 *Answer: tilings by rows, tiling by columns and one tiling by quadrants.*

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0

0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1
0	1	1	1	0	0	0	1

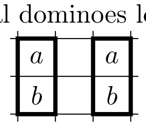
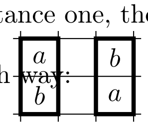
0	0	0	0	1	1	1	1
0	0	0	0	1	1	1	1
0	0	0	0	1	1	1	1
0	0	0	**0**	**1**	1	1	1
1	1	1	**1**	**0**	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0

We consider two cases: whether a square with units on the side diagonal (bold) appears in the tiling or not. *It appears.* First, the "cross"of width 2 with the center in this square is uniquely filled, then one by one the remaining cells are filled and we get the tiling by the quadrants.

It does not. Any 2×2-square is either monochrome, or the boundary between zeroes and ones divides a square into two dominoes. If the tiling is not monochrome, then there are neighbours on the side 0 and 1. Further, the strip of width 2 containing them is uniquely filled: it is either two monochrome rows or columns. In the first case, it turns out that all the horizontals are monochrome; in the second - that all verticals are monochrome.

B4 *Answer:* $6(2^n + 2^m - 4)$.

In two vertical dominoes located at distance one, the letters can be located

in such way:  or in such way: 

We call a line (horizontal or vertical) *striped*, if in it two types of letters alternate. It is easy to see that if a line is striped, then all the lines parallel to it are also striped.

c	d	c	d	c	d	c	d	c	d	c	d
a	b	a	b	a	b	a	b	a	b	a	b

We show that there is at least one striped line. Suppose the top row of the table is not striped, then it has different letters (let a and b) at a distance of 2.

			a		b		
			b		a		
			a		b		
			b		a		

Then it is clear that these cells are in striped columns.

There are $6 \cdot 2^n$ tilings with striped columns and $6 \cdot 2^m$ tilings with striped rows. 24 of them are counted twice.

B5 *Answer: 7.*

There are 16 squares 2×2. They are divided into 7 groups, and in order to prohibit 7 periodic tilings, it is necessary to ban at least one square from each group (we illustrate for each group a periodic tiling which would be permitted no square of the group were forbidden).

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

1	1
1	1

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

0	0
0	0

1	1	1	1	1	1
0	0	0	0	0	0
1	1	1	1	1	1
0	0	0	0	0	0
1	1	1	1	1	1
0	0	0	0	0	0

1	1
0	0

0	0
1	1

1	0	1	0	1	0
1	0	1	0	1	0
1	0	1	0	1	0
1	0	1	0	1	0
1	0	1	0	1	0
1	0	1	0	1	0

1	0
1	0

0	1
0	1

1	1	1	1	1	1
1	0	1	0	1	0
1	1	1	1	1	1
1	0	1	0	1	0
1	1	1	1	1	1
1	0	1	0	1	0

1	0
1	1

0	1
1	1

1	1
0	1

1	1
1	0

0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0

1	0
0	1

0	1
1	0

0	0	0	0	0	0
1	0	1	0	1	0
0	0	0	0	0	0
1	0	1	0	1	0
0	0	0	0	0	0
1	0	1	0	1	0

1	0
0	0

0	0
1	0

0	0
0	1

0	1
0	0

Example of 7 forbidden squares:

1	1
1	1

0	0
0	0

1	0
1	0

1	1
0	0

1	0
0	1

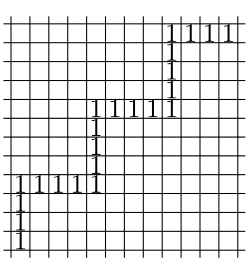
1	0
0	0

1	1
0	1

The 4 last local rules force that there is no zero below a one. Together with the first rule, this gives that ones cannot be in adjacent columns, and together with the second – that zeroes cannot be in adjacent columns. Then the only possible tiling is a periodic tiling with alternating columns, but the third local rule also prohibits it.

B6 Let $N \times N$ be the size of the largest local rule. It's clear that if we fill the entire plane with zeroes and draw several occurrences of the allowed picture so that the distance between any two of them is greater than N, then we obtain a permitted tiling.

B7 A *ladder* is an infinite stepped figure of units, we have drawn a fragment of the ladder with step 4 (all empty cells are filled with zeros).



If N is the size of the largest local rule, then all ladders with a step greater than N are permitted, because all the $N \times N$ patterns already appear in some large frame, so they are permitted.

Since a ladder is not a union of frames, it is not a beautiful tiling.

B8 At first glance it seems that there is nothing to prove here, since the plane is, in some way, a circle of infinite radius. But in fact, from the fact that we know how to cover *arbitrarily large* circle it does not follow automatically that we can cover an *infinitely large* circle.

Extra problem about types of infinity. Kashchei the Immortal (russian folklore villain) uses a magical currency exchange. It allows to change one gem for any positive integer number of gold coins, or one gold coin for any number of silver coins, or one silver coins for any number of copper coins. Initially, Kashchei has one hundred of gems. To maintain his immortality, Kashchei has to spend one copper coin per day Suppose he has no any other income. Will he be able to live for a billion of years? A googol of years? For eternity?

Let's return to the problem. If we add a few more tiles to a tiling A of some finite region and obtain a tiling B, we say that the tiling A *can be extended* to B.
If we find an infinite sequence of tilings A_1, A_2, A_3 such that each previous tiling can be extended to the next one, and for any k the tiling A_k covers a circle of radius k with center in $(0, 0)$, then we win – we take the union of these tilings.

We call a tiling of a finite region *good* if for any R it can be extended to a tiling, covering a disc of radius R with center at zero. It suffices to show that a good tiling can be extended for any R to a good tiling that covers a circle of radius R.

Let D be a good tiling. Consider all possible ways to extend it to a tiling that covers a circle of radius R. There is a finite number of such extensions; if none of these extensions is a good tiling, then for each of them there is a radius to which you cannot tile. We take the maximum of these radii and find that the original tiling is also not good.

B9 It follows from the tasks of the following sections.

B10 It follows from the tasks of the following sections..

B11

1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	0	**1**	**1**	0	**1**	**1**	0	**1**
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**1**	0	0	0	**1**	**1**	**1**
1	0	**1**	0	0	0	**1**	0	**1**
1	**1**	**1**	0	0	0	**1**	**1**	**1**
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	0	**1**	**1**	0	**1**	**1**	0	**1**
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

a) We give a solution that uses some facts of set theory, namely, that the continuum is greater than the countable set. We denote the substitution by the letter s. Note that $s^k(1)$ is a square of size $3^k \times 3^k$, consisting of eight squares $s^{k-1}(1)$ and one square $s^{k-1}(0)$; all the squares of form $s^k(0)$ are filled with zeroes.

Any $3^k \times 3^k$-tiling of form $s^k(1)$ can be extended to the tiling $s^{k+1}(1)$ in eight ways; each extension corresponds to one of the eight external cells of the square 3×3 and different shifts of the square $3^{k+1} \times 3^{k+1}$ are obtained. Notice that we are interested in squares in the plane, not only in abstract squares.

We start with the square A_1 of size 1×1, in which there is a one, and we will do one of such extensions at each step; the filled area will expand. We obtain a sequence of tilings $A_1, A_2, A_3, \ldots$

Such sequences correspond to sequences of digits from 1 to 8, there is continuum of such sequences. The union of squares A_i can be either a quarter-plane, or a half-plane, or the whole plane. If we did not obtain the entire plane, then, starting at some point, we expanded the squares at only one direction. It is clear that there are uncountably many sequences $\{A_i\}$, the union of which fills the entire plane. The resulting tilings of the plane will be called *limit tilings*.

We show that different sequences $\{A_i\}$ lead to different limit tilings (note that we do not claim that they are not equivalent).

We consider in limit tiling squares $(3^k + 2) \times (3^k + 2)$, at the boundary of which there are ones, and zeroes inside. It is clear that the centers of all such squares are shifted relatively to the center of the square A_k by a vector, both coordinates of which are multiples of 3^k. Therefore, knowing the limit tiling, we can determine the coordinates of the center A^k modulo 3^k. But it is easy to see that the coordinates of its center cannot differ from the coordinates of the center of A_1 by more than $3^k/2$, therefore two limit tilings that coincide determine the same sequence of squares.

It is easy to see that each limit tiling is uniquely decodable, therefore there is at least a continuum of various Sierpinski carpets. And since only a countable number of carpets can be equivalent to one another, the number of equivalence classes is infinite.

b) The periodic Sierpinski carpet is one that consists of only zeroes. We show that all the remaining Sierpinski carpets are non-periodic. We note that if there is at least one 1 in the Sierpinski carpet, then for any natural number k the carpet contains the square $s^k(1)$, in the center of which there is a square of size $3^{k-1} \times 3^{k-1}$ filled with zeroes, and this square is framed by units. If we consider a translation by some nonzero vector whose coordinates are less than 3^{k-1}, the image of the border will intersect the square of zeroes, and so for any k. So, there are no vectors of periodicity.

c) *Answer: no.*

Let n be the size of the largest local rule; take k such that $3^k > n$. Consider a tiling A of the plane by 1's. It is not decodable, so the tiling $s^k(A)$ is not infinitely decodable. We show that the tiling $s^k(A)$ is permitted.

$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$

a pattern of $s^k(A)$

$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(0)$	$s^k(1)$	$s^k(1)$	$s^k(0)$	$s^k(1)$	$s^k(1)$	$s^k(0)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(0)$	$s^k(0)$	$s^k(0)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(0)$	$s^k(1)$	$s^k(0)$	$s^k(0)$	$s^k(0)$	$s^k(1)$	$s^k(0)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(0)$	$s^k(0)$	$s^k(0)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$
$s^k(1)$	$s^k(0)$	$s^k(1)$	$s^k(1)$	$s^k(0)$	$s^k(1)$	$s^k(1)$	$s^k(0)$	$s^k(1)$
$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$	$s^k(1)$

$s^{k+1}(1)$

It can be shown that any square with side no more than n that occurs in $s^k(A)$, occurs in some block composed of four squares $s^k(1)$. But this block is located inside $s^{k+1}(1)$, and therefore it is a pattern of Sierpinski carpets and does not contain forbidden squares.

B12 Let n be the size of σ. The tiling $\sigma(A)$ is composed of $n \times n$-squares – images of letters. Any 2×2-square is located inside some $2n \times 2n$-square (union of 4 such squares), which is of the form $\sigma\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$. Since the square $\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}$ is permitted and σ agrees with local rules, $\sigma\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$ does not contain forbidden patterns.

B13 Consider a tile with letter a and a sequence of squares

$$A_0 = a, A_1 = \sigma(a), A_2 = \sigma^2(a) \dots$$

In this sequence A_i has size $n^i \times n^i$, and it follows from the previous problem that A_i is permitted for any i. From **B8**, it follows that there exists a permitted tiling of the whole plane.

B14 *Answer: not always.* For alphabet $\{0, 1\}$ and substitution $\sigma(0) = \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$, $\sigma(1) = \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ such a tiling does not exist.

The criterion for the existence of a fixed tiling. σ admits a fixed tiling if and only if one letter appears strictly inside its own image, or two letters appear in their own images in some opposite side of the square, strictly, or four letters appear in their own images in all possible corners.

Sketch of proof. Let σ be a $k \times k$-substitution. First note that requiring $S = \sigma(S)$ would involve the choice of an origin tile, which is not a problem here: we just want S and $\sigma(S)$ to be equivalent. This means that there exist integers i, j such that, in S, the image by σ of any cell (x, y) is the pattern appearing at positions $(i + kx \cdots i + kx + k - 1, j + ky \cdots j + ky + k - 1)$. We can do a euclidean division of i by $1 - k$, and get $i = (1 - k)x + i'$ for some i' with $0 \le i' < k - 1$. This x has the property that $i + kx \le x < i + kx + k - 2$. Similarly, there is some y such that $j + ky \le y < j + ky + k - 2$. We get that the image of the letter a which appears in cell (x, y) of S contains the letter a itself. A little more precisely, the inequalities give that the letter appears in its own image, but not on the bottom or right boundary (because we have divided by $k - 1$ rather than k). It it appears in the left boundary, then $x = i + kx$, so that the image of cell $(x - 1, y)$ is

$(x+k-1 \cdots x, j+ky \cdots j+ky+k-1)$: again the letter at cell $(x-1,y)$ appears in its own image. The same is true if (x,y) was in the top boundary of the square.

Conversely, suppose σ admits one letter a which appears stricly inside its own image, say at position i,j. Then $\sigma^t(a)$ can be seen as a square pattern of size $k^t \times k^t$, that we can index in $-k^t i/(k-1) \cdots k^t(k-1-i)/(k-1)$ (and similary for the vertical coordinate) so that a is in cell $0,0$. This square grows on all four sides, so that, ultimately, every pair (x,y) of integers is mapped to a letter for these large enough squares. Moreover, this letter is constant, once defined. Indeed, an easy induction allows to show that then the square $\sigma^t(a)$ appears in the middle of any $\sigma^{t'}(a)$, for $t' \geq t$. Thus we can define S as holding, in cell (x,y) the letter that appears as $\sigma^t(a)(x,y)$ for large enough t. By construction $\sigma(S)$ will be the shift of S (with respect to i,j).

If σ admits two letters that appear in their own images in some side of the square, strictly, or four letters that appear in their own images in all possible corners, then we can do the same kind of iterations, starting with the pattern consisting in these two or four letters.

C. Examples of non periodicity

C1 a) For example, such a substitution σ:

$$\boxed{1} \longrightarrow \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 1&0&1\\ \hline 1&1&1\\ \hline\end{array} \qquad\qquad \boxed{0} \longrightarrow \begin{array}{|c|c|c|}\hline 0&0&0\\ \hline 0&1&0\\ \hline 0&0&0\\ \hline\end{array}$$

The fact that the set of infinitely decoded tilings cannot be defined by local rules is proved in the same way as for Sierpinski carpets.

We call such a figure (and also if we swap 0 and 1) *a double frame.*

	1	1	1	1	1	1	1	
	1	0	0	0	0	0	1	
	1	0				0	1	
	1	0				0	1	
	1	0				0	1	
	1	0	0	0	0	0	1	
	1	1	1	1	1	1	1	

We say that the *size* of the double frame in the figure above is 5. If a double frame of size N occurs in the tiling, then the tiling cannot have a vector of periodicity with absolute values of coordinates less than N. By induction on k, it is shown that $\sigma^k(0)$ and $\sigma^k(1)$ contain a double frame of size 3^{k-1}.

b) Consider a one-letter alphabet.

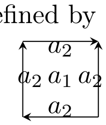
C2 Suppose that in there are 8 letters a_i, b_i, c_i, d_i, for $i = 1$ or 2. The images of any letter is of form $\begin{smallmatrix} a_i & b_j \\ c_k & d_l \end{smallmatrix}$. There are 16 ways to choose indices, so we can choose different images for all eight letters.

C3 It is a consequence of some following problems.

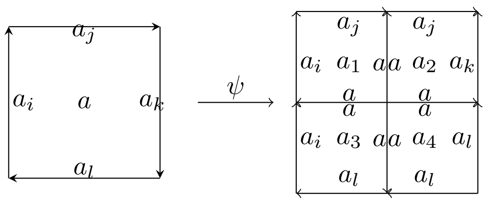
C4 Let us prove the aperiodicity of any infinitely decodable tiling A. Firstly note that, from the separation images property, it follows that both coordinates of any periodicity vector v must be divisible by 2, and then note that the vector $v/2$ is a periodicity vector for the tiling $\sigma^{-1}(A)$. Applying this to a tiling with the smallest nonzero period, we have a contradiction.

C5 We use notation W for left direction, E for right, N for up, and U for down. We will denote by NW the left-up direction (and similar notations for other diagonal directions). Our substitution has the image separation property, so each letter a_i of the alphabet A belongs to one of the types NW, NE, SW, SE.

We draw letters of the new (decorated) alphabet B as tiles, in the center of each tile a letter from A is written, there are arrows on the sides and on each side has a *label* – a letter of A. Thus, a letter from B is defined by a 5-tuple of letters of A and four extra bits (arrow directions).



We will talk about types of tiles (i.e. NE, NW, SE or SW, types of central letters), types of tile sides (N, E, S, W depending on the direction of the arrow), types of side labels (NE, NW, SE, SW). The substitution $\psi : B \to B^4$ is defined as follows. If the letter a is written in the center of a tile and some labels are written on the edges, and $\sigma(a) = \begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}$, then we write the letters a_1, a_2, a_3, a_4 in centers of coresponding tiles, preserve labels and directions for outer sides of tiles, direct central arrows from the center of 2×2-square and draw labels a on them, see pic.:

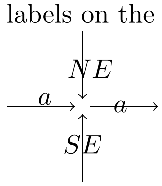


The idea of such a decoration is as follows: an infinitely decoded tiling consists of blocks $2^k \times 2^k$, and each $2^k \times 2^k-$block consists of four $2^{k-1} \times 2^{k-1}-$blocks, and boundaries between these four blocks form a cross, on the edges of which the same labels are written. These crosses separate blocks of large size from each other. In the Sierpinski carpets example, there was no such boundary, and local rules could not distinguish the boundary of a large block from its middle part.

We say that the direction NE is symmetric to NW with respect to the vertical direction, and SW is symmetric to SE with respect to the vertical direction. Symmetry relative to the horizontal direction is defined similarly.

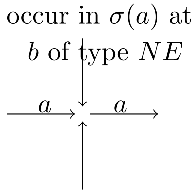
We define the following local rules:

1) On neighbouring tiles, adjacent sides have the same direction and label (that is, one can speak about the direction and label of edges in the entire tiling).
2) Only tiles of vertically symmetrical types can have a common vertical side. Only tiles of horizontally symmetrical types can have a common horizontal side.
3) Nodes between tiles can have only such degrees of: outgoing degree 4 OR indegree 3 and outdegree 1.
4) If 4 edges go from one node, their labels coincide.
5) If there is a tile of type NW to the up-left side of a given node, we call this node *central*. We require that central nodes have outgoing degree 4. If these edges have labels a, and $\sigma(a) = \begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}$, then in the center of the tile to the north-west of the node there should be the letter a_1, similarly for the other three tiles.
6) From a node of type 3-1 there is an *outgoing arrow*, *central ingoing arrow* and two *lateral arrows*. We require that the labels on the central incoming and outgoing arrows coincide, and also that the types of labels on the lateral arrows are symmetrical with respect to the outgoing edge. For example:



7) For a node of type 3-1, suppose there is a label a on the outgoing v_1, and a label b on a lateral ingoing arrow v_2. Let $\sigma(a) = \begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}$.

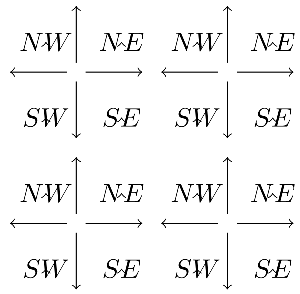
   If the angle between the type of b and the direction of v_2 is equal to $135°$ (for instance, v_2 goes down, and b has type NE), then the angle between v_1 and the type of b must equal to $45°$, and the letter b must occur in $\sigma(a)$ at the position corresponding to the type of b.



$b = a_2$, where a_2 is the NE letter in $\sigma(a)$

To construct this set of rules, we looked closely at an infinitely decodable tiling, and forbid everything that did not occur there. It is necessary, on the one hand, to write enough rules so that decoding can be determined, and on the other hand – to check that after the decoding all the local properties hold. For this reason, we do not use larger local rules (involving non-neighbouring tiles): it is difficult to verify them after decoding.

Analysis of permitted tilings. Now we want to investigate permitted tilings. From 1) we find that the plane is divided into 2×2-squares, in each of which the tiles have the form $\begin{smallmatrix} NW & NE \\ SW & SE \end{smallmatrix}$. We will call these blocks *basic blocks*. From 5) and 4) it follows that the centers of the basic blocks are central nodes; moreover each of them has 4 outgoing arrows, which have the same labels.

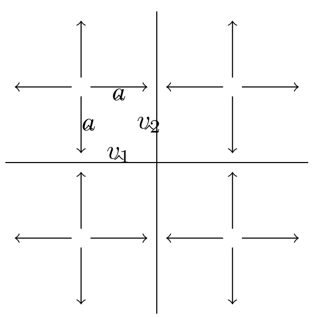


All the nodes are divided into central, *lateral* (those which have 2 ingoing arrows at the picture above) and the rest. From 3) it follows that the midpoints of the sides of the basic blocks (lateral nodes) are of the type $3-1$, and together with 6) it implies that the arrows on each side of the basic block are co-directional and have the same labels.

Together with property 4) this gives that each base block is a $\psi-$ image of some letter. Thus, the decoding from a permitted tiling T to $\psi^{-1}(T)$ is well defined. This tiling can be obtained from T by erasing the lines inside the base blocks and writing inside them the letters which are the labels of arrows, outgoing from its center. We need to check for $\psi^{-1}(T)$ the properties of $1)-7)$. Properties 1), 3), 4), and 6) are automatically satisfied, since no new situation at the nodes (in terms of which labels and arrows are written on the incoming edges) arises.

For property 2), we must check that any two base blocks having a common side are decoded into symmetric type of letters. But the type after decoding coincides with the type of the label written on the inner cross of the basic block. The local rule 5) guarantees that the marks of the two blocks are symmetric, and after decoding, 2) holds. So, the basic blocks are combined into 4×4-blocks, which after decoding have the form $\begin{smallmatrix} NW & NE \\ SW & SE. \end{smallmatrix}$

Consider a 4-tuple like in the following figure.



The label a has type NW, therefore, according to 6), the edge v_1 is directed to the left, and v_2 to the top. Hence, v_1 and v_2 go out from a central node. From 4) it follows that the labels on these edges coincide. Also, from 6) it follows that a is in the upper left corner of the image of this label under σ. Similarly we can speak about the rest of the basic blocks, and therefore, after decoding, property 5) holds. So, the decoded tiling is permitted, and therefore all the permitted tilings are infinitely decoded.

Generally speaking, we have not solved the problem yet. We have not shown that *each* tiling that is infinitely decodable by σ allows a permitted decorations. It can be shown that such a decoration exists for those tilings for which each tile lies strictly inside some block of form $\sigma^k(a_i)$. You can describe the rest of the tilings and slightly modify our decoration; we leave this it to the reader as an exercise.

C6 We leave this task as an exercise.

C7 We'll use $C5$. Let σ be a substitution $A\to A^4$. Take τ 2×2 with property of separating images $B\to B^4$. Construct a new alphabet C of size $|A|\cdot|B|$; its letters are pairs $(a_i,b_j)|a_i\in A, b_j\in B$. We define on it a substitution ψ of size 2×2 given as follows: $(a,b)\to \begin{smallmatrix}(a_i,b_j) & (a_k,b_l)\\ (a_m,b_n) & (a_p,b_r)\end{smallmatrix}$, where $\sigma(a)=\begin{smallmatrix}a_i & a_k\\ a_m & a_p\end{smallmatrix}$ и $\tau(b)=\begin{smallmatrix}b_j & b_l\\ b_n & b_r\end{smallmatrix}$. It is easy to see that it is a decoration of σ and it has the property of separating images.

D. Different formalisms.

D1 Replace letters at up and right sides by opposite ones.

D2 *Hint.* Use Robinson tilings, divide them by small squares and give local rules that force small tiles to form big Robinson tiles.

D3 a) *Answer: no.* It's easy to construct a periodic tiling repeating a 2×2 block made of one tile and all its reflections.

b) *Answer: yes.*

E1. a) alphabet $\{a_1, a_2, b\}$; forbid $\{ba_1, a_2b, bb, a_2a_1\}$.

b) alphabet $\{a_1, a_2, b\}$; forbid $\{ba_1, a_2b, bb, a_2a_1\}$ horizontally, $\{\begin{smallmatrix} b \\ a_2 \end{smallmatrix}, \begin{smallmatrix} a_1 \\ b \end{smallmatrix}, \begin{smallmatrix} b \\ b \end{smallmatrix}, \begin{smallmatrix} a_2 \\ a_1 \end{smallmatrix}\}$ vertically, and $\{\begin{smallmatrix} a_1 & a_2 \\ a_2 & \end{smallmatrix}\}$.

E2. The set of tiles with decorations, without those labeled b, tiles arbitrarily large squares (in every infinite tiling). By **B8**, there is a tiling of the plane using those tiles. It is labeled by only a's.

E3. a) *Answer: no.* Assume it could, and let k be the maximal size of the local rules. Consider a tiling with a single connected component which is a horizontal path of size k, and another tiling, similar but with size $k+1$. These two tilings have the same patterns of size k, so they should be either both forbidden or both permitted. Nevertheless, exactly one of these two is in the wanted set.
b) alphabet $\{a, b_1, b_2\}$; forbid $\{ab_2, b_1b_1, b_2b_2, b_1a\}$.
c) One possible solution (among many possibilities): Decorations draw a spanning forest for the connected component, that is the disjoint union of rooted directed trees that cover the whole component. Each tree should have even cardinality: this can be counted at each node by addition modulo 2 of the value carried by possible son nodes. On the one hand, every connected component can be decorated by an even spanning tree, hence permitted. On the other hand, if a tiling is permitted, every connected component is spanned by several trees of even cardinality, so that it itself has even cardinality.

E4*. In 1D: yes, easy: alphabet $\{a, b_1, b_2\}$; forbid $\{ab_2, b_1b_1, b_2b_2, b_2a\}$.

In 2D: no, it's much more difficult, because the sum of two odd numbers is not odd. It's hard to force the connected component to be spanned by only one tree (rather than a forest). This question is currently the purpose of ongoing research, on so-called *soficity* of sets of tilings.

E5. The set of tilings representing 3-colorable planar graphs, for instance.

E6.
a) All 2×2 patterns that appear in the new tiling already appeared in one of the two previous permitted tilings.
b) Let k bet the cardinality of the decorated alphabet. The number of possible decorations for the border of $n \times n$ tilings is $k^{4(n-1)}$.
c) Consider the set of configurations such that all c's form a column, and consider the set of $n \times n$-squares that appear on the left of this column. They can be any tiling over $\{a, b\}$, so there are 2^{n^2} different ones. If n is big with respect to k, then by the previous question, two of these tilings have the same decorations within the corresponding infinite tilings, and by Question a), we can replace one by the other. But then (at least) one of the two does not have the correct mirror image on the other side of the column.
d) If $\mathcal{S}$ is a set of tilings that is defineable by decorations and local rules, then there exists k such that, for every n, any set of $n \times n$ tilings which have nonintersecting pairwise extension sets (set of possible ways to extend as a tiling of the plane) has cardinality at most k^n.