Remarkable points of polygons

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It is well-known that any triangle has many various, more or less remarkable points. For instance, the number of so-called *triangle centers* in the encyclopedia [1] has already exceeded 10000. These points define various lines, circles and other objects related to the triangle. **The aim of this project (our pipe dream)** is to find analogues of these objects in an arbitrary polygon. Of course, it will be realized in the least, but on the other hand the participants of the Conference will have a large scope for their own research.

1 Barycenters of polygons

The centroid of a triangle is the point M of concurrence of its medians. Indeed, if we place equal weights in the vertices of a triangle, their barycenter M_0 will coincide with M. The same point is the barycenter M_2 of a triangle cut out from cardboard for example. However the barycenter of a triangle made of wire differs from these points. Denote it by M_1. It can be determined using the following general property of barycenters.

The main property. Suppose some figure F is a disjoint union of figures F' and F''. Then the barycenter M of F lies on the segment $M'M''$, where M', M'' are the barycenters of F' and F'' respectively. Moreover, the ratio MM'/MM'' equals the ratio m_2/m_1 of masses of F'' and F'. Here, if the figures F' and F'' are piecewise linear curves, then the weight of a figure is proportional to its length, and for plane figures the weights are proportional to their areas.

1.1. Find the barycenter M_1 of a triangle made of wire.

1.2. Prove that the point M_1, the point M_0 of concurrence of medians and the incenter I of ABC are collinear (the **Nagel line**), moreover M_0 divides IM_1 in the ratio $2:1$.

1.3. Prove that M_1 is the radical center of three excircles of ABC, i.e., the segments of tangents from M_1 to these circles are equal.

1.4. Prove that each of the lines A_0M_1, B_0M_1, C_0M_1, where A_0, B_0, C_0 are the middle points of the segments BC, CA, AB respectively, bisects the perimeter of ABC.

So, for any triangle we can define two barycenters M_0 and M_1, and they are somehow connected to each other. An arbitrary polygon can have three barycenters: the barycenter M_0 of its vertices, the barycenter M_1 of the union of its sides and the barycenter M_2 of the whole polygon (in the degenerate case of a triangle we have $M_2 = M_0$).

1.5. Determine points M_0 and M_2 for a quadrilateral $ABCD$.

1.6. Prove that M_0 lies on the segment LM_2, where L is the intersection point of the diagonals of the quadrilateral, and M_0 divides LM_2 in the ratio $3:1$.

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1.7. Determine point M_1 for a quadrilateral $ABCD$.

It seems that generally the point M_1 has no remarkable properties. However for a circumscribed $ABCD$ the situation is different.

1.8.

a) Show that M_2 lies on the segment IM_1, where I is the incenter, and divides it in the ratio $2:1$.

b) Show that the same is true for any circumscribed polygon.

1.9. Show that in any quadrilateral, M_0 is the midpoint of the segment M_1W, where W is the midpoint of IL.

Now consider a quadrilateral which is not only circumscribed, but inscribed as well. Poncelet theorem then asserts that one can fix the incircle and the circumcircle and "rotate" the quadrilateral between them.

1.10. What is the locus of each barycenter of the quadrilateral?

Appendix. Definition of barycenters

It is not obligatory to deliver the solutions of exercises given here, but each delivered solution gives a bonus to the participant, namely the right to tell the solution of one problem of the project orally.

Definition 0. Material point is a pair (X, m), where X is a point of the plane, and m is a positive number ("the mass" of the point).

Definition 1. The mass center of material points $(X_1, m_1), \dots, (X_n, m_n)$ is the point M such that

$$m_1\overrightarrow{MX_1} + \dots + m_n\overrightarrow{MX_n} = \overrightarrow{0}.$$

Exercise 1. Prove that there exists a unique mass center.

Exercise 2. Prove that for any point O

$$\overrightarrow{OM} = \frac{m_1\overrightarrow{OX_1} + \dots + m_n\overrightarrow{OX_n}}{m_1 + \dots + m_n}.$$

Exercise 3. Prove that the mass center of material points (A, m_1) and (B, m_2) lies on the segment AB and dissects it in the ratio $m_2 : m_1$.

Exercise 4. Prove that the mass center of material points $(A, 1)$, $(B, 1)$, $(C, 1)$ coincides with the centroid of triangle ABC.

Exercise 5. Prove that the mass center of material points (X_1, m_1),..., (X_n, m_n), (X_{n+1}, m_{n+1}) coincides with the mass center of material points $(M, m_1 + \dots + m_n)$, (X_{n+1}, m_{n+1}), where M is the mass center of (X_1, m_1),..., (X_n, m_n).

Definition 2. The mass center of n segments having at most one common point pairwise is the mass center of material points $(M_1, l_1), \dots, (M_n, l_n)$, where M_i is the midpoint of i-th segment, and l_i is its length.

Definition 3. Let F be the union of n triangles having no common inner points pairwise. **The mass center** of F is the mass center of material points $(M_1, S_1), \dots, (M_n, S_n)$, where M_i is the centroid of i-th triangle, and S_i is its area.

Exercise 6. Prove that for any division of a polygon into triangles, the mass center of the union of these triangles is the same point (called the **mass center of the polygon**).

Exercise 7*. Let lines AB and CD meet at point E, and lines AD and BC meet at point F. Prove that the midpoints of segments AC, BD and EF are on a line (**the Gauss line** of quadrilateral $ABCD$).

In this project the barycenter M_0 of a polygon $A_1 \ldots A_n$ denotes the mass center of material points $(A_1, 1), \ldots, (A_n, 1)$, the barycenter M_1 denotes the mass center of segments $A_1A_2, \ldots, A_{n-1}A_n, A_nA_1$, and the barycenter M_2 denotes the mass center of the polygon.

2 Euler and Nagel lines

It is well known that in any triangle the circumcenter O, the centroid M_0 and the orthocenter H lie on a line, which is called **Euler line**. Moreover M_0 divides the segment OH in the ratio $1:2$. Furthermore let A_1, B_1, C_1 be the points of tangency of the incircle with the sides BC, CA, AB respectively, and the points A_2, B_2, C_2 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the corresponding sides (these are the points of tangency of the sides with the corresponding excircles). Then the lines AA_2, BB_2, CC_2 concur at the point N which is called **Nagel point**. One can show that M_0 lies on the segment IN and divides it in the ratio $1:2$. Also note that each of the lines AA_2, BB_2, CC_2 divides the perimeter of the triangle in two equal parts. Our goal is to find analogues of the Euler line for an inscribed polygon and the Nagel line for a circumscribed one.

2.1. (A. Myakishev, II Sharygin olimpiad) Let $ABCD$ be an inscribed quadrilateral and O be its circumcenter. Let H_a, H_b, H_c, H_d be the orthocenters of triangles BCD, CDA, DAB, ABC respectively, and H be the intersection point of the lines H_aH_c and H_bH_d. Show that the barycenter M_2 lies on the segment OH and divides it in the ratio $1:2$.

2.2. (A. Myakishev, II Sharygin olympiad) Let $ABCD$ be a circumscribed quadrilateral, and I be its incenter. Let T, U, V, W be the points which are symmetric to the points of tangency of the incircle with the sides AB, BC, CD, DA respectively with respect to their midpoints.

a) Show that any of lines TV and UW divides the perimeter of the quadrilateral into two equal parts.

b) Let N be the point of intersection of lines TV and UW. Prove that M_2 lies on the segment IN and divides it in the ratio $1:2$.

Another approach to the definition of the Euler line was proposed by I.Romanov [4].

Define the orthocenter of an inscribed n-gon $A_1 \dots A_n$ inductively. Let $H_1, \dots, H_n$ be the orthocenters of $(n-1)$-gons $A_2 \dots A_n$,...,$A_1 \dots A_{n-1}$ respectively.

2.3. Prove that the lines $A_1H_1, \dots, A_nH_n$ are concurrent.

2.4. Let us call the corresponding intersection point H the orthocenter of the n-gon. Show that the barycenter M_0 lies on the segment OH and divides it in the ratio $(n-2):2$.

2.5. Let $ABCD$ be an arbitrary quadrilateral.

Consider two generalizations of the orthocenter:

H^* is the center of the parallelogram formed by the orthocenters of triangles ABL, BCL, CDL, DAL;

$H^{**} = H_aH_c \cap H_bH_d$, where as usual H_a is the orthocenter of triangle BCD, and so on.

Furthermore we generalize O as $O^{**} = O_aO_c \cap O_bO_d$ where O_a is the circumcenter of triangle BCD, and so on (in other words, O^{**} is the intersection of perpendicular bisectors to AC and BD).

Prove that

a) M_0 is the midpoint of $O^{**}H^*$;

b) (Ya. Ganin, A. Myakishev) M_2 lies on the segment $O^{**}H^{**}$ and divides it as $1:2$;

c) H^* is the midpoint of LH^{**}.

3 Quasi-centers of the circumcircle and the incircle

In this section we will try to define the points O and I for an arbitrary quadrilateral, which have the properties similar to those of the circumcenter and the incenter. Of course, for an inscribed (resp. circumscribed) quadrilateral the point O (resp. I) should coincide with the center of the circumcircle (resp. the incircle).

3.1. Show that for any quadrilateral $ABCD$ which is both inscribed and circumscribed, the centers O, I and the intersection point L of diagonals are collinear.

3.2. Let I be the intersection point of the diagonals of a quadrilateral $PQRS$. Denote the projections of I to PQ, QR, RS, SP by A, B, C, D respectively. Show that

a) the quadrilateral $ABCD$ is circumscribed iff the quadrilateral $PQRS$ is inscribed;

b) if the quadrilateral $ABCD$ is circumscribed then I is its incenter.

Let A', B', C', D' be the intersection points of lines IA, IB, IC, ID with RS, SP, PQ, QR respectively.

3.3. Prove that

a) the quadrilateral $ABCD$ is inscribed iff $PR \perp QS$;

b) if $PR \perp QS$, then $A'B'C'D'$ is a rectangle and the points A, B, C, D, A', B', C', D' are concyclic.

3.4. Construct a quadrilateral $PQRS$ by the points A, B, C, D if it is known that I lies inside $ABCD$.

Definition. Define the **quasi-incenter** of a convex quadrilateral $ABCD$ to be the point I constructed in the previous problem, and the **quasi-circumcenter** to be the intersection point O of the lines $A'C'$ and $B'D'$. (We assume that I lies inside $ABCD$)

3.5. Show that the quasi-centers O, I and the intersection point L of the diagonals are collinear.

3.6. For a quadrilateral which is both inscribed and circumscribed express the circumradius and the inradius in terms of lengths of the segments OI and OL.

The last problem enables us to define the quasi-incircle and the quasi-circumcircle for an arbitrary quadrilateral. Up to date it is unknown whether these circles have any interesting properties.

Now let us describe another approach to defining quasi-centers.

3.7. Let I_a, I_b, I_c be the excenters of the triangle ABC, and J be the circumcenter of $I_aI_bI_c$. Prove that O is the midpoint of the segment IJ.

3.8. Prove that for an arbitrary quadrilateral, its internal angle bisectors form an inscribed quadrilateral, and so do the external angle bisectors.

Denote by I and J the circumcenters of the quadrilaterals formed by the internal angle bisectors and the external angle bisectors of $ABCD$ respectively.

3.9. (VII Sharygin olimpiad) Show that for an inscribed quadrilateral with circumcenter O, the points I and J are symmetric with respect to O.

Now we can take I and the midpoint O of the segment IJ to be the quasi-incenter and the quasi-circumcenter. Unfortunately, with this definition the intersection point L of the diagonals can be not contained in the line OI.

One more approach to defining the quasi-circumcenter is proposed in [6].

3.10. Let X be the intersection point of lines AB and CD, Y be the intersection point of AD and BC, Z be the intersection point of AC and BD. Let M_X be the Miquel point of lines AD, BC, AC and BD, $M_Y$be the Miquel point of AB, BD, AC и BD, and M_Z be the Miquel point of AD, BC, AB и CD. Prove that

a) the lines XM_X, YM_Y and ZM_Z are concurrent;

b) if the points A, B, C, D are concyclic then these lines intersect at the circumcenter of $ABCD$.

The obtained point can also be considered as a quasi-circumcenter.

4 Additional problems

4.1. Let $ABCD$ be a quadrilateral without parallel sidelines circumscribed around a circle centered at I. The sides AB, BC, CD, DA touche the incircle at points X, Y, Z, T respectively. As usually $L = AC \cap BD$ (also $L = XZ \cap YT$). Let X' be the reflection of X about the midpoint M_{AB} of side AB; Y', Z', T' are defined similarly; $N = X'Z' \cap Y'T'$ is the Nagel point.

Prove that the condition $M_0 = I$ is equivalent to each of the following conditions:

a) $AX + CZ = BY + DT$;

b) $XZ \parallel X'Z'$ (или $XZ \parallel M_{AB}M_{CD}$);

c) X', Z' and $BC \cap AD$ are collinear;

d) L, I, N a collinear;

e) (A.Zaslavsky, M.Isaev, D.Tsvetov, All-Russian olympiad 2005 г.) $IA \cdot IC = IB \cdot ID$.

4.2. (A.Myakishev) Triangles ABC and $A'B'C'$ are called **ortologic**, if the perpendiculars from A', B', C' to BC, CA, AB respectively concur. Quadrilaterals $ABCD$ and $A'B'C'D'$ are called ortologic, if the triangles ABC and $A'B'C'$, BCD and $B'C'D'$, CDA and $C'D'A'$, DAB and $D'A'B'$ are ortologic. Let $ABCD$ and $A'B'C'D'$ be ortologic, AC and BD meet at L, $A'C'$ and $B'D'$ meet at L'. Prove that $AL : LC = A'L' : L'C'$ and $BL : LD = B'L' : L'D'$ (i.e. ortologic quadrilaterals are affine equivalent).

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