# Remarkable points of polygons
## Solutions

## 1 Barycenters of polygons

1.1. **Answer.** The incenter of the triangle $A_0B_0C_0$.

**Proof.** Note that the points $A_0$, $B_0$, $C_0$ are the midpoints of the segments $BC$, $CA$, $AB$, so let us place the weights equal to the lengths of these segments in this points. Then the barycenter of the weights in the points $A_0$ and $B_0$ is the point, which divides the segment $A_0B_0$ in the ratio $AC : BC = A_0C_0 : B_0C_0$, i.e. the base of the angle bisector in $A_0B_0C_0$, therefore the barycenter of all three weights lies on this bisector. Analogously, we deduce that it lies on the other angle bisectors of $A_0B_0C_0$ too, hence it coincides with, its incenter (fig.1).

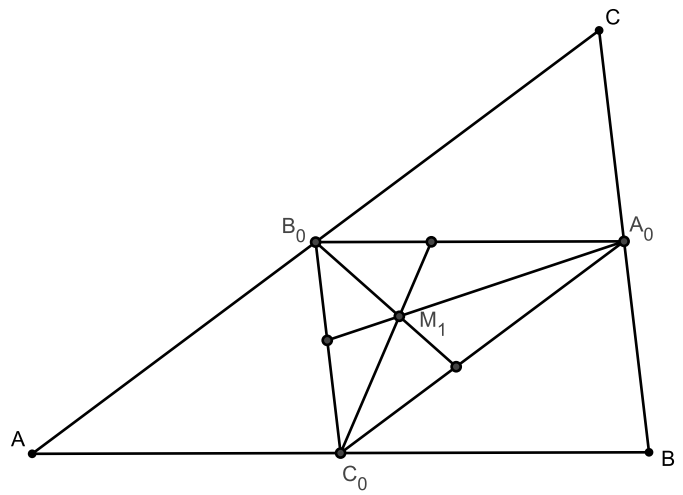


Fig. 1

1.2. It follows from the previous problem and the fact that the triangles $ABC$ and $A_0B_0C_0$ are homothetic with the center at $I$.

1.3. Denote by $X$ and $Y$ the points of tangency of excircles, which are tangent to the segments $AC$ and $BC$, with the line $AB$. We have $BX = AY = p$ (where $p$

is the semiperimeter of the triangle), hence $C_0X = C_0Y$. Moreover, the line passing through the centers of these circles is orthogonal to the angle bisector of $C$, hence, it is also orthogonal to the line $C_0M_1$. THus we obtain that $M_1$ lies on the radical axis of these teo circles. Arguing in the same way we deduce that $M_1$ lies on the radical axis of any pair of the three excircles.

1.4. Let $X$ be the intersection of $C_0M_1$ with the segment $AC$. It follows from the previous problem that $X$ is the midpoint of the segment with the ends at the tangency points of the line $AC$ with the excircles, which are tangent to the segments $AC$ and $BC$ (fig.2). As the distances from these points to $A$ are equal to $p - c$ and $p$ respectively, we obtain that $AX = p - c/2$ and $AX + AC_0 = p$.

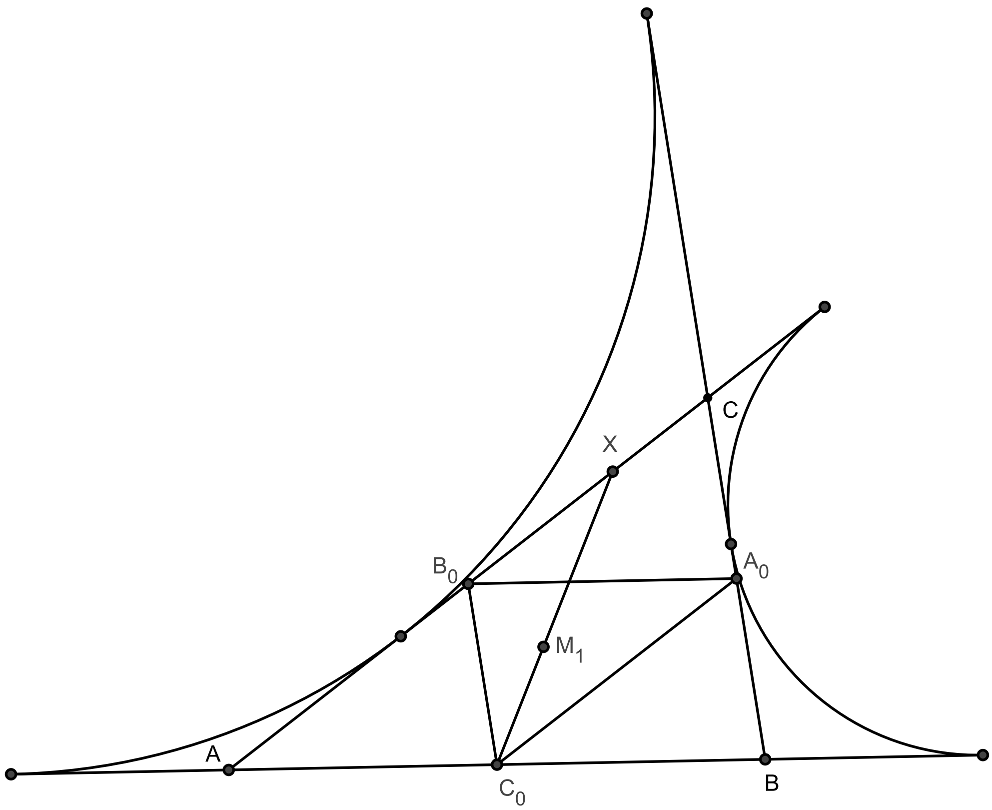


Fig. 2

1.5. **Answer.** $M_0$ is the center of the parallelogram $PQRS$, where $P$, $Q$, $R$, $S$are the midpoints of $AB$, $BC$, $CD$, $DA$. $M_2$is the intersection point of the line $l_1$, passing through the centroids of $ABC$ and $ADC$, and the line $l_2$, passing through the centroids of $ABD$ and $BCD$.

1.6. Let $U$, $V$ be the midpoints of the diagonals $AC$ and $BD$ respectively and $L$ be their intersection point. The centroid of $ABC$ lies on its median $BU$ and divides it in the ratio $2:1$. Analogously, the centroid of $ACD$ lies on $DU$ and divides it in the same ratio. The line $l_1$, passing through these centroids, is parallel to $BD$ and it intersects $AC$ at the point, which divides the segment $UL$ in the ratio $1:2$. In the same way we obtain that $M_2$ lies on the line, which is parallel to $AC$ and which divides the segment $VL$ in the ratio $1:2$. Note that the lines passing through $M_0$ parallel to $AC$ and $BD$ pass through the midpoints of the segments $UL$ and $VL$ respectively. The assertion of the problem now follows straightforwardly (fig.3).

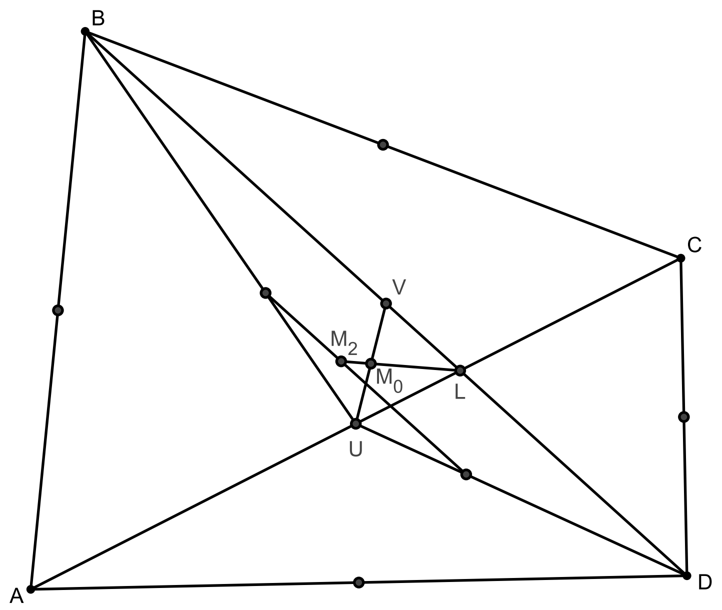


Fig. 3

1.7. As $P$ and $Q$ are the barycenters of the segments $AB$ and $BC$, we obtain that the barycenter $X_1$ of their union lies on $PQ$ and $PX_1/QX_1 = BC/AB$. This point mat be constructed in the following way: construct the angle bisector $BB'$ of the triangle $BPQ$ and take a point, which is symmetric to $B'$ with respect to the midpoint of $PQ$. Analogously one can construct the barycenters $X_2$, $Y_1$, $Y_2$ of the piecewise linear curves $CDA$, $DAB$, $BCD$. Then $M_1$is the intersection point of the lines $X_1X_2$ and $Y_1Y_2$ (fig.4).

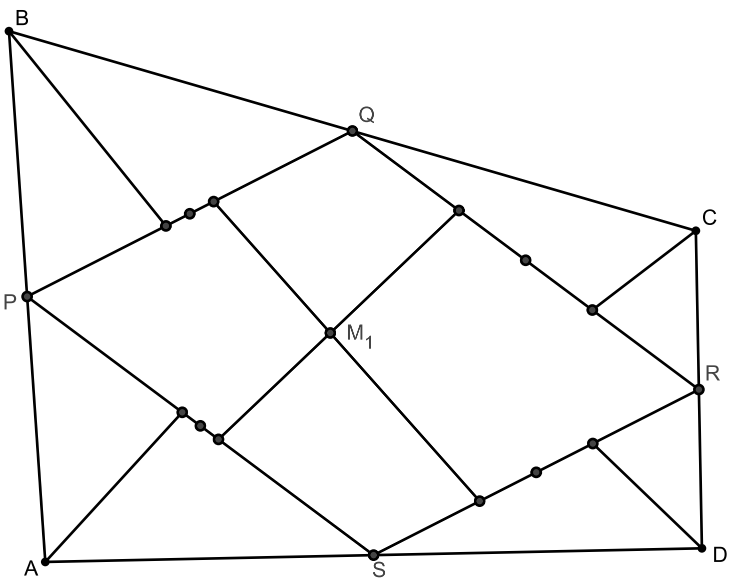


Fig. 4

1.8.

a) Let $I$ be the incenter. The centroids of the (solid) triangles $IAB$, $IBC$, $ICD$, $IDA$ divide their medians $IP$, $IQ$, $IR$, $IS$ in the ratio $2:1$, i.e. the quadrilateral, which is formed by them, is homothetical to $PQRS$ with the center in $I$ and the coefficient $\frac{2}{3}$. As the ratio of the areas of $IAB$, $IBC$, $ICD$, $IDA$ is the same as the ratio of the corresponding sides of $ABCD$, this homothety maps $M_1$ to $M_2$.

b) The proof is analogous to the previous one.

1.9. It follows from the previous problems and the theorem about three homothety centers.

1.10. **Hint.** $M_0$ is the midpoint of the segment, formed by the midpoints of the diagonals of the quadrilateral. Note that the Gauss line of the circumscribed quadrilateral contains its incenter. This fact implies that the locus of $M_0$ is a circle. Now one can use the corresponding homotheties to deduce that the loci of two other centers are also circles (fig.5).

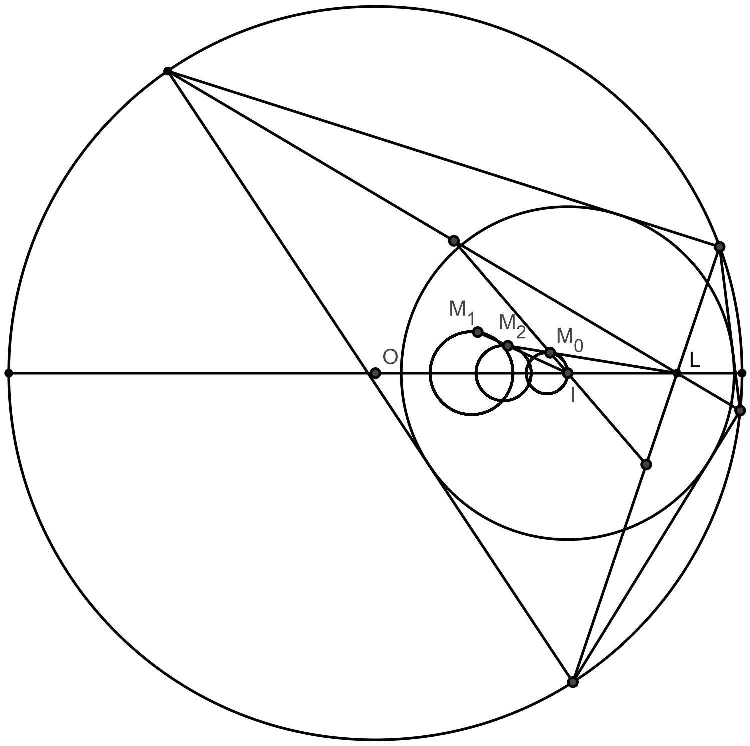


Fig. 5

## 2 Euler and Nagel points

2.1. Let $M_a$ and $H_a$ be the centroid (the point $M_2$) and the orthocenter of $BCD$ respectively. Denote the centroids and the orthocenters of the other triangles analogously. All there triangle share a common circumcircle, whose center is $O$. By considering the Euler lines of these triangles, we note that the quadrilateral $M_aM_bM_cM_d$ is mapped to the quadrilateral $H_aH_bH_cH_d$ under the homothety with the center in $O$ and the coefficient 3, so the intersection points of the corresponding diagonals are mapped to each other analogously.

2.2.

a) It follows from the definition that $AT + DV = BC$, $BT + CV = AD$, i.e. $TA + AD + DV = VC + CB + BT$.

b) Let $a$, $b$, $c$, $d$ be the length of tangent segments to the incircle from the vertices $A$, $B$, $C$, $D$. It is clear that if one places the weights $a$, $b$, $c$, $d$ into the points $A$, $B$, $C$, $D$ respectively, then the barycenter of the obtained system is the point $N$, and if one places the weights $2a+b+d$, $2b+a+c$, $2c+b+d$, $2d+c+a$ into those vertices, then the barycenter is the point $M_1$. It remains to show that $I$ is the barycenter of the weights $b+d$, $a+c$, $b+d$, $a+c$ and apply the assertion of problem 1.8..

The point $I$ satisfies $S_{IAB} - S_{IBC} + S_{ICD} - S_{IDA} = 0$. The same property holds for the midpoints $U$ and $V$ of the diagonals of the quadrilateral, therefore these points are concurrent (this fact is called *the Monge theorem*). Now let $X$ and $Y$ be the tangency points of the incircle with the sides $BC$ and $AD$ respectively. Then the angles formed by the line $XY$ and these sides are equal and also $XY$ passes through the intersection point of the diagonals $L$ by Brianchon's theorem. Applying the laws of sines to the triangles $LXB$ and $LYD$, we obtain that $BL/DL = b/d$. Similarly, $AL/CL = a/c$. These identities and the equalities $S_{UBC}/S_{UAD} = BL/DL$, $S_{VBC}/S_{VAD} = CL/AL$, $S_{IBC}/S_{IAD} = (b+c)/(a+d)$ imply that $I$ divides the segment $AC$ in the ratio $(a+c)/(b+d)$, QED.

2.3. **Hint.** Use induction.

2.4. **Hint.** Use induction.

2.5. a) Let $M_{AC}$, $M_{BD}$ be the midpoints of diagonals $AC$, $BD$. It is clear that $O^{**}M_{AC} \perp AC$ and $O^{**}M_{BD} \perp BD$. On the other hand $H^*M_{AC} \perp BD$ and $H^*M_{BD} \perp AC$ because $H^*$ is the center of the parallelogram having the sidelines perpendicular to the diagonals of the given quadrilateral and passing through its vertices. Hence $H^*M_{AC}O^{**}M_{BD}$ is a parallelogram which yields the required assertion.

b) **Hint.** Use the assertion of problem 4.2.

c) Clearly follows from two previous assertions.

## 3 Quasi-centres of the circumcircle and the incircle

3.1. We use the following fact.

**Lemma.** Suppose the point $P$ lies inside the circle with the center at $O$. Denote by $X$ and $Y$ the intersection points of the circle with two orthogonal half-lines with the initial point at $P$. Then the locus of the intersection points of the tangent lines to the circle at $X$ и $Y$ is a circle with the center, lying on $OP$.

**Proof.** Let $Z$be the fourth vertex of the rectangle $PXZY$. We have $OP^2+OZ^2 = OX^2+OY^2$, thus the locus of $Z$ is the circle with the center at $O$. Hence the locus of the midpoint of the segment $XY$ is a circle with the center at the midpoint of $OP$, so the locus of the intersection of the tangent lines, which is inverse to it, is also a circle.

Now the assertion follows from the fact that the lines, passing through the tangency points of the opposite sides of the quadrilateral with the incircle, are orthogonal and they both pass through the intersection point of its diagonals.

3.2. As the quadrilaterals $IAQB$ and $IBRC$ are inscribed, we have $\angle IBA = \angle IQA$, $\angle IBC = \angle IRC$, which imply both assertions.

3.3. The proof is similar to the previous problem.

3.4. Suppose that the half-lines $PQ$ and $SR$ intersect at $X$. Then we have $\angle PXS = \angle PIS - \angle IPA - \angle CSI$. The quadrilaterals $IAPD$ and $ICSD$ are inscribed, thus we obtain that $\angle IPA + \angle CSI = \angle CDA$. Besides, $\angle PIS = \angle RIQ = (\angle PAD + \angle DCS + \angle RCB + \angle BAQ)/2 = (\angle ABC + \angle CDA)/2$. Therefore, $\angle AIC = \pi - \angle PXS = \pi - (\angle ABC - \angle CDA)/2$. Arguing in the same way we can find the angle $BID$ and thus to construct $I$ as the intersection point of the corresponding arcs of the circles, and then we are able to construct the quadrilateral (fig.6).

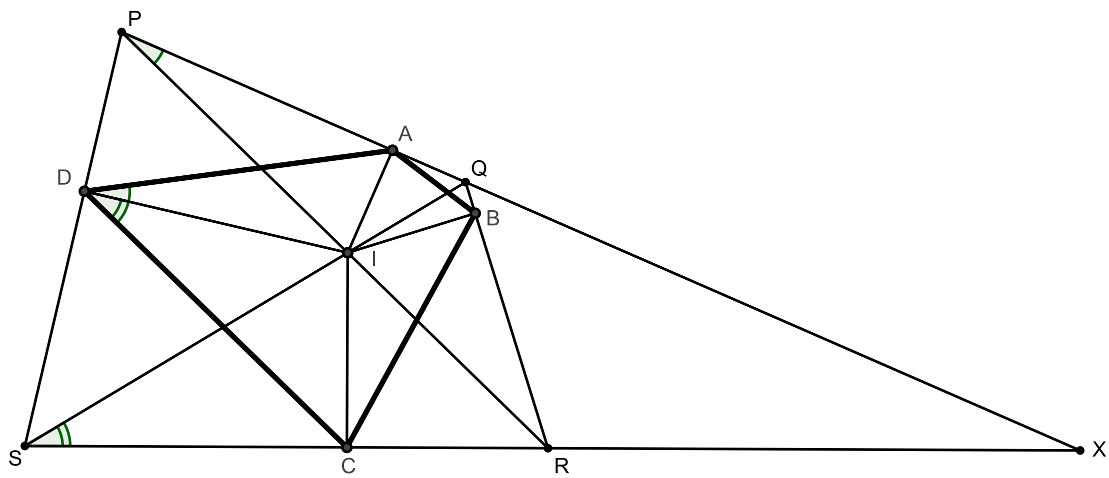

Fig. 6

3.5. **Hint.** Apply the central projection, which maps the points $P$, $Q$, $R$, $S$ to the vertices of a parallelogram.

3.6. **Answer.** $R^2 = \frac{OL \cdot OI^2}{2OI - OL}$, $r^2 = \frac{(R^2 - OI^2)^2}{2(R^2 + OI^2)}$.

**Hint.** It follows from Poncelet theorem that it is enough to consider the quadrilateral having a diameter of the circumcircle as one of the diagonals.

3.7. **Hint.** Prove that the segments $I_aJ$, $I_bJ$, $I_cJ$ are orthogonal to the corresponding sides of the triangle $ABC$.

3.8. One can prove this by a direct computation of angles.

3.9. Let the angle bisectors of $A$ and $B$ intersect at $K$, the angle bisectors of $B$ and $C$ — at $L$, the angle bisectors of $C$ and $D$ — at $M$, the angle bisectors of $D$ and $A$ — at $N$ (pic.9.4). Then the line $KM$ is the bisector of the angle, formed by $AD$ and $BC$. Denote this angle by $\phi$, exterior angle theorem implies that $\angle LKM = \angle B/2 - \phi/2 = (\pi - \angle A)/2 = \angle C/2$,hence $\angle LIM = \angle C$. On the other side, the line through $L$ orthogonal to $BC$ and the line through $M$ orthogonal to $CD$ form with $ML$ the angles, equal to $(\pi - \angle C)/2$, i.e. the triangle, formed by these lines and $ML$ is isosceles and its vertex angle is equal to $C$. Thus its vertex coincides with $I$. Therefore the lines through the vertices of $KLMN$ orthogonal to the respective sides of $ABCD$ pass through $I$ (fig.7). Similarly we obtain that the perpendiculars from the vertices of the quadrilateral formed by the exterior angle bisectors pass through $J$.

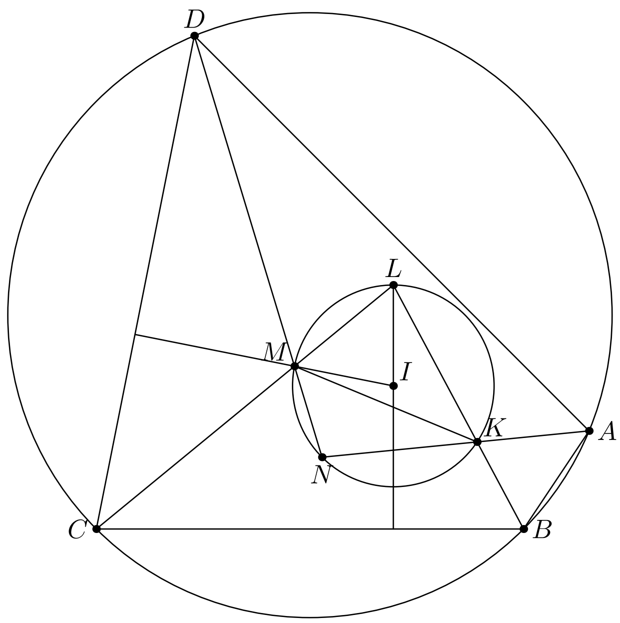


Fig. 7

Now let $K'$be the intersection points of the exterior bisectors of $A$ and $B$. As the quadrilateral $AKBK'$ is inscribed in a circle with the diameter $KK'$, we obtain that the projections of $K$ and $K'$ to $AB$ are symmetric with respect to the midpoint of $AB$. From this assertion and the one, proved above, it follows that the projections of

$I$ and $J$ to each of the sides of $ABCD$ are symmetric with respect to the midpoints of the corresponding sides, which is equivalent to the assertion of the problem.

3.10. See [6].

## Список литературы


[1] http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[2] Ф.Ивлев. Центры тяжести многоугольников. Доклад на ММКШ. 2008. https://www.mccme.ru/circles/oim/mmks/notes.htm

[3] A.Akopyan. Some remarks on the circumcenter of mass. https://arxiv.org/pdf/1512.08655.pdf

[4] И.Романов. Прямая Эйлера $n$-угольника. Доклад на ММКШ. 2017. https://www.mccme.ru/circles/oim/mmks/works2017/ignatov2.pdf

[5] А.Заславский. Диагонально-перпендикулярное отображение четырехугольников. Квант. 1998. №4.

[6] M.Rolnek, Le Anh Dung. The Miquel Points, Pseudocircumcenter, and Euler-Poncelet Point of a Complete Quadrilateral. Forum Geometricorum V.14 (2014). https://personal.us.es/rbarroso/trianguloscabri/sol/FG201413.pdf.