Quarks game

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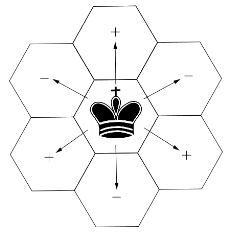
1. The plane is divided into cells in the form of regular hexagons. A King can move to any neighbor cell, see the picture. Moves up, right-down and left-down will be called *positive* and the rest — *negative*. The King made several moves and returned to the original cell. What are possible values for the difference between the number of positive and negative moves?



This simple mathematical problem has fundamental physical consequence: *the charge of any free particle is a multiple of the electron charge.* Want to see how school mathematics helps to understand particle physics? Read on! Knowledge of physics is not required, we are going to give all the necessary information. Let us start with an informal introduction; mathematical definitions appear after Problem 9.

Elementary particles and where they live[1]

Substance consists of atoms, atoms consist of nuclea and electrons, nuclea consist of protons and neutrons.

All that are *particles* of substance. Particles can be either *elementary* or *composed*, i.e., consisting of other particles. E.g., a proton is composed from two up-quarks and one down-quark, while the quarks are elementary; see the figure. For our purposes it is not important *what* are quarks; just imagine sticky balls glued together to form other particles.

-proton-

If a particle is separated from any other particles (i.e., moves independently on them) in an experiment, then the particle is called *free*. E.g., normally a neutron is contained in a nucleus, but it can be made free in nuclear reactions. This collection of problems is about *confinement of quarks* (main problems are 11, 27, 33.b):

Quarks cannot be free.

What this actually means is absolutely paradoxical: quarks exist but cannot be observed in an experiment in principle. We are going to discuss both reasons and consequences of this (so far unproved) assertion[2].

Each particle has a *charge*. E.g., an electron has charge -1 (this is how the measurement units are chosen), and an up-quark has charge $+2/3$. The charge of a composed particle equals to the sum of charges of the contents. E.g., a proton has charge $+1$.

Each particle has an *antiparticle* having opposite charge. E.g., an up-antiquark has charge $-2/3$.

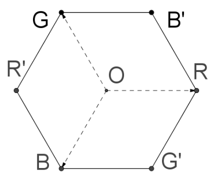
The list of just a few known quarks and free composed particles is given in the table.

Particle				Antiparticle			
Name	Notation	Content	Charge	Name	Notation	Content	Charge
Elementary particles: quarks (to the left) and antiquarks (to the right)							
up-quark	u	-	$+2/3$	up-antiquark	$\bar{u}$	-	$-2/3$
down-quark	d	-	$-1/3$	down-antiquark	$\bar{d}$	-	$+1/3$
Composite particles from a quark and an antiquark							
pion	π^+	$u\bar{d}$	$+1$	pion	π^-	$\bar{u}d$	-1
Composite particles from 3 quarks (to the left) or 3 antiquarks (to the right)							
proton	p	uud	$+1$	antiproton	$\bar{p}$	$\bar{u}\bar{u}\bar{d}$	?
neutron	n	?	0	antineutron	$\bar{n}$	?	?

2. Complete the entries with a questionmark in the table.

[1] This is only an exposition of a commonly accepted theory and does *not* pretend to be the final truth about particles.
[2] Actually the proof of a related assertion is one of the Millenium problems, the essence of which we shall also explain.

Each particle has also a *color* (it has nothing to do with light). It is not a number, but a vector in the plane. Hereafter we assume that $RB'GR'BG'$ is a regular hexagon with the center at the origin O. A quark can have one of the colors $\overrightarrow{OR}$, $\overrightarrow{OG}$, $\overrightarrow{OB}$ ("red", "green", "blue"). Thus there actually 3 distinct up-quarks and 3 down-quarks.



The color of an antiparticle is opposite to the color of a particle. E.g., the antiparticle of a red up-quark has color $\overrightarrow{OR'}$ ("cyan"). The color of a composed particle is the vector sum of colors of the components. E.g., a proton consisting of a red up-quark, a green up-quark, and a blue down-quark, has vanishing color. Actually the reason for confinement of quarks is *confinement of color*:

Particles with nonvanishing color cannot be free.

As a warm-up, let us discuss basic consequences of confinement of color.

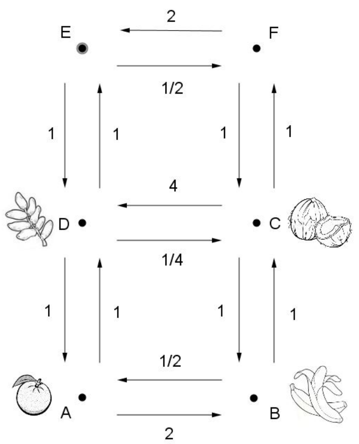
3. a) For which q and p satisfying $p+q \leqslant 3$, there is a particle with vanishing color consisting of q quarks and p antiquarks[3]?

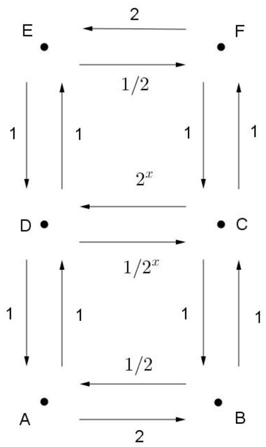
b) Prove that each particle with vanishing color consisting of quarks and antiquarks from the above table has integer charge.

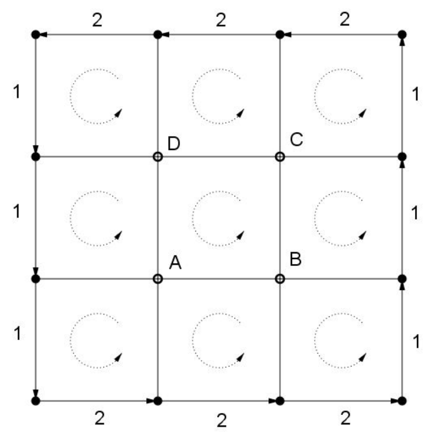
Now start the way to understanding the reasons of confinement of color. We shall need a theory describing interaction of quarks. All known interactions except gravitation are described by *gauge theory*. This concerns not only interaction of quarks but also everyday-observed effects like magnetic interaction of conductors with currents. As we shall see now, the *idea* of gauge theory is very simple.

Toy model of gauge theory

Several cities are connected by roads in the shape of an $M \times N$ grid; see the figure. Each city has its own type of goods (in unlimited quantity). E.g., city A has apples and city B has bananas. For two neighboring cities A and B an exchange rate $U(AB) > 0$ is fixed, e.g., 2 bananas for an apple. The rate is symmetric, i.e., $U(BA) = U(AB)^{-1}$: one gets back an apple for 2 bananas.







A cunning citizen can travel and exchange along a 1×1 square $ABCD$ to multiply his initial amount of goods by a factor of $U(AB)U(BC)U(CD)U(DA)$. E.g., in the figure to the left the factor is $2 \cdot 1 \cdot 4 \cdot 1 = 8$. Denote $U(ABCD) := U(AB)U(BC)U(CD)U(DA)$.

We shall see that the *logarithms* of these factors are more convenient quantities. All we need to know about logarithms is the following definition. If $y = 2^x$ for some real numbers x and y, then x is called the *logarithm* of y and is denoted by $x = \log_2 y$. E.g., $\log_2 2 = 1$, $\log_2 \sqrt{2} = \frac{1}{2}$, $\log_2 1 = 0$, $\log_2 \frac{1}{2} = -1$.

In particular, a trip along the square $ABCD$ gives profit in one of the directions, if $\log_2 U(ABCD) \neq 0$. The *total speculation profit* is measured by the sum $S[U]$ of the values $\log_2^2 U(ABCD)$ over all 1×1 squares $ABCD$ (say, bypassed counterclockwise). E.g., in the figure to the left

$$S[U] = \log_2^2 U(ABCD) + \log_2^2 U(DCFE) = (\log_2 8)^2 + (\log_2 \tfrac{1}{2})^2 = 10.$$

You are the king, who can set exchange rates except those on the boundary of the grid. You set them to minimize the quantity $S[U]$. The resulting rates are called *optimal*.

4. Clean up the kingdom in the left figure, i.e., find the number x for which the total speculation profit in the middle figure is minimal.

[3]There can be more quarks: particles from 4 quarks or antiquarks were discovered in 2014, from 5 — in 2015.

Optimal exchange rates can be found approximately on the computer, scrolling with a small step all possible rates from a certain interval and comparing the corresponding values of $S[U]$.

5. Do it for the kingdoms in the figures in the middle and to the right. Is there any faster algorithm?

Notice that the change of variables $x(AB) := \log_2 U(AB)$ simplifies the expression for the profit a lot:

$$S[x] := \sum_{\substack{\text{all } 1\times 1 \text{ squares } ABCD \\ \text{bypassed counterclockwise}}} (x(AB) + x(BC) + x(CD) + x(DA))^2.$$

For each 1×1 square $ABCD$ denote $x(ABCD) := x(AB) + x(BC) + x(CD) + x(DA)$. E.g., in the figure to the left, $x(ABCD) = 3$. Denote by W the factor multiplying the initial amount of goods for a counterclockwise travel around the whole boundary. E.g., in the figure to the left, $W = 4$.

6. Assume that the fixed rates at the boundary are as in the figure, i.e.,

$$U(AB) = \begin{cases} 2, & \text{if } AB \text{ is on the southern or northern border of the grid} \\ & \text{and is directed counterclockwise along the boundary;} \\ 1, & \text{if } AB \text{ is on the eastern or western border of the grid.} \end{cases}$$

Clean up the kingdom for the following particular grid sizes, i.e., complete the table[4]:

Grid	1×2	1×3	$1 \times N$	2×2
The value W	4			
The minimal value for $S[U]$				
Optimal rates $U(AB)$ for all roads AB				
Values $x(AB)$ for all roads AB				
Values $x(ABCD)$ for all 1×1 squares				

7. a) For which values of the boundary rates in an $M \times N$ grid one can achieve $S[U] = 0$?
b)* Assume that $S[U] = 0$. Can the citizen get a profit by moving along a closed path?
c) For which values of M and N the optimal rates are uniquely determined by the boundary rates?
d) How M, N, W and the minimal speculation profit are related?
e) Which of the grids — 8×8 or 7×9 — has smaller speculation profit, if W is the same?

8. Now we are allowed to change boundary rates. What is the minimal possible value of the expression

$$\frac{1}{2} S[U] - j \log_2 W$$

that we can obtain for a fixed real number j on the $M \times N$ grid?

Physical interpretation

Our toy model allows to describe roughly magnetic interaction of conductors with currents.

Let j be a real number. The grid boundary is a frame with a current strength j. Current in the frame creates a magnetic field. The value $x(ABCD)$ at a 1×1 square $ABCD$ represents the *magnetic flux* through the square, the sum $S[x]$ represents the *magnetic field energy*, and $\frac{1}{2}S[U] - j \log_2 W$ represents the energy of the whole system. Each system tries to minimize its energy. Thus the values $x(AB)$ are set to minimize the expression $\frac{1}{2}S[U] - j \log_2 W$. Beware: the model is very rough!

If the positions of the conductors are not fixed, i.e., the grid size is not fixed, then the system tries to move the conductors to decrease the energy. This means attraction or repulsion of conductors. Movement of conductors is understood as change of the number of cells, but not the sizes of the cells themselves.

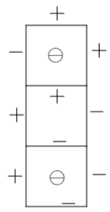
9. a) Does a frame with current tend to reduce or increase its area?
b) Do parallel conductors with opposite currents magnetically attract or repulse?
c)* And if the current directions are the same?
Hint. Close the conductors far away to obtain a frame (for b)) or two frames (for c)).

[4]The team with the maximal number of completed entries at the semifinal will receive the same number of apples.

The quark interaction is described by *quantum* gauge theory. The main difference is that the exchange rates are *random* and not necessarily positive. For simplicity let us assume that the exchange rates take only two values $+1$ and -1, shortly: "$+$" and "$-$". Bypassing the boundary of the square changes the amount of goods, if there is an odd number of signs "$-$" on its boundary.

Now it is time for precise mathematical definitions and theorems. First we state the modified model, then a pair of problems, and only after that introduce all the required definitions little by little.

Toy model for quantum gauge theory



To each road AB of the $1 \times N$ grid in the figure, randomly assign a sign "$+$" or "$-$" (so that all possible assignments of signs have equal probability; see the definition of *probability* before Problem 12). For each assignment U let us denote by $S'[U]$ the number of squares 1×1 having an odd number of signs "$-$" on the sides. E.g., $S'[U] = 2$ in the figure. The expectation of the random variable $S'[U]$ is called the *energy $E(N)$ of electromagnetic interaction between a quark and an antiquark at distance N.* (See the definitions of a *random variable* and its *expectation* before and after Problem 17.)

Remark. The latter term is unsplittable; the words 'quark', 'antiquark', 'electromagnetic interaction' have no separate formal meaning. Beware: the model is very rough!

10. Find the energy $E(N)$ for the grids: 1×1; 1×2; 1×3; $1 \times N$;
a) approximately using a computer simulation; b) exactly.

11. (Quarks confinement in 1-dimensional space.) Does there exist a number E_0 such that $E(N) \leqslant E_0$ for all N? (The same informally: is the amount of energy required to move a quark and an antiquark far away from each other finite or infinite?)

Classical probability[5]

It is useful to start learning probability theory on the "physical" level of rigor as in books [Shen], [KZhP]. Here we give "mathematical" definitions from the beginning. However many problems are stated using "practical" language and we show how to formalize some of them. The formalization of the remaining problems is left to the reader. Such formalization is the first step of the solution, on which the answer can depend.

Consider an experiment that has m equally probable outcomes, such as rolling a dice, drawing a card out of a deck, etc. If the event in question (for example, the fall of the six, drawing an Ace, etc.) occurs in a outcomes, then the *probability* of the event is said to be $p = a/m$.

This explanation is useful for beginners, but it is not a mathematically rigorous definition. Here is the mathematical definition.

The *probability* of a subset A of a finite set M is the following number

$$P(A) = P_M(A) := |A|/|M|.$$

Hereafter, unless otherwise stated, the set M is fixed and omitted from the notation. Then the probability is defined for all its subsets. They are often called *events*.

12. From the deck of 52 cards one card is drawn. Find the probability that it is
(a) of black suit; (b) an Ace; (c) with a picture;
(d) Queen of spades; (e) a King or a diamond.

E.g. in the problem 12 (c) the set M ("the set of all possible events") coincides with the set of all cards in the deck, the set A ("the set of events that we consider") is the set of all cards with pictures. This is the way to formalize this problem (and many other probabilistic problems).

13. A coin is tossed 3 times. Find the probability of getting
(a) three Heads; (b) two Heads and one Tail.

[5]This section is taken from [EMZ] with small changes.

To solve some of the above-stated problems, the following rules are useful.

14. (a) *Addition rule.* Let $A \cap B = \emptyset$. Express $P(A \cup B)$ in terms of $P(A)$ and $P(B)$.
(b) Express the probability $P(A \cup B)$ in terms of $P(A)$, $P(B)$ and $P(A \cap B)$.
(c) *Multiplication Rule.* Express the probability $P_{M\times N}(A \times B)$ in terms of $P_M(A)$ and $P_N(B)$.

15. a) Signs "+" and "−" are randomly assigned to the 4 sides of a square. Find the probability that the number of signs "−" is odd.
b) What is the probability that for the 1×2 grid we have $S'[U] = 0$? $S'[U] = 1$? $S'[U] = 2$?

The following problem leads to the notions of a *random variable* and the *expectation*.

16. a) I suggest you the following game. You pay 2 candies, then a die is rolled, and you get as many candies as there occur points on the die. Is the game profitable for you?
b) Now the rules are the same except that you pay 100 candies, when 1 point occurs. (You have enough candies to pay.) Is the game profitable for you?
c) A bank promises you a reliable profit absolutely for free. You put your 8 candies into the bank, then a die is rolled. If 2, 3, or 4 points occur, then you get back your contibution of 8 candies plus 1 candy in addition. If 5 or 6 points occur ("market growth"), then you get even plus 2 candies in addition to your contribution. But if 1 point occurs ("crysis"), then you loose the whole your contribution ("We are sorry but we cannot do anything for you — that's crysis.") Is the game profitable for you?

A real-valued function X defined on the set M is called a *random variable.* The set of pairs $(x_i, p_i), i = 1, 2, \ldots$, where $\{x_1, x_2, \ldots\}$ is the set of all possible values of the random variable X, and $p_i = P(\{m \in M\colon X(m) = x_i\})$, $i = 1, 2, \ldots$, are the corresponding probabilities, is called the *distribution* of the random variable X. Further, we denote by $X = x_i$ the event $\{m \in M\colon X(m) = x_i\}$.

17. Find the distribution of the random variable $S'[U]$ for the grid 1×2.

The mathematical expectation (or *mean-value*) of the random variable X is the sum

$$\mathrm{E}(X) := \sum x_i p_i = x_1 P(X = x_1) + x_2 P(X = x_2) + \ldots.$$

18. Find the expectation of the random variable $S'[U]$ for the grid 1×2.

Write $+1$ instead of "+" and -1 instead of "−". Let the random variable X_k be the product of all numbers on the sides of the k-th square from the top in the $1 \times N$ grid and the random variable W' be the product of all numbers on the boundary of the grid. (Similarly we define W' for the $M \times N$ grid.)

19. Express $S'[U]$ and W' in terms of X_1 and X_2 for the grid 1×2. Find $\mathrm{E}(X_1)$, $\mathrm{E}(X_2)$ and $\mathrm{E}(W')$.

20. (a) Prove that the expectation of the random variable X defined on M is equal to $\sum\limits_{m\in M} X(m)/|M|$.
(b) Prove that if $\mathrm{E}(X) \leqslant x$, then there exists $m \in M\colon X(m) \leqslant x$.
(c) A random variable X takes the same value μ for all $m \in M$. Find $\mathrm{E}(X)$.
d) Let a, b be real numbers and X, Y be random variables. Is the following equality always true $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$?
(e) And what about $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$?

Now let us proceed to a more accurate model of quark interaction (further we work in this model).

Solid model for quantum gauge theory

Fix a number $c > 1$ called the *interaction constant.* To each road AB of the $M \times N$ grid, randomly assign a sign "+" or "−" so that an assignment U has the probability

$$P[U] = \frac{c^{-S'[U]}}{\sum\limits_{\text{all the assignments } V} c^{-S'[V]}}.$$

The *energy of the electromagnetic interaction between a quark and an antiquark at distance* N is

$$E_{M,c}(N) = -\frac{1}{M}\log_2 \mathrm{E}(W').$$

Our aim is to find it. (Here $\mathrm{E}(W')$ is the expectation of the random variable W', defined before the problem 18. We do not know any easy explanation of the formula.)

Statistical probability

The following more general definition of the probability is used in this model. Consider a set M and for each $m \in M$ assign a non-negative number $P(m)$, so that the sum of all such numbers is equal to 1. Then the *probability* of the event A is the sum of all $P(m)$ over all $m \in A$.

21. a) Find the probability of each assignment of signs for the 1×1 grid. Compute $\mathrm{E}(W')$ and $E_{1,2}(1)$.
b) The same, if one considers only the assignments such that all roads except the top one have sign "+".

22. How the probability of an assignment changes, if the signs of all roads coming from a city are changed?

The following definition generalizes the multiplication rule from 14 (c). First, assume that the probabilities $P(m)$ are the same for all elements $m \in M$. Two subsets (or events) A and $B \neq \emptyset$ of the finite set M are called *independent*, if the probability of the set $A \cap B$ in B is equal to the probability of the set A in M. Let us give a symmetric reformulation, which works for $B = \emptyset$ and for more general definitions of the probability, when not all probabilites $P(m)$ are equal. The subsets A and B of the finite set M are called *independent*, if

$$P(A \cap B) = P(A) \cdot P(B).$$

The main example of the independent subsets is the following. In the set of all cells of the chessboard the subset of cells in first three rows and the subset of cells in last four columns are independent.

23. Are the following subsets independent, if all elements of M have equal probabilities?
(a) The subsets $\{1, 2\} \subset \{1, 2, 3, 4\}$ and $\{1, 3\} \subset \{1, 2, 3, 4\} = M$.
(b) The subsets $\{1, 2\} \subset \{1, 2, 3, 4, 5, 6\}$ and $\{1, 3\} \subset \{1, 2, 3, 4, 5, 6\} = M$.

Random variables X and Y are called *independent*, if the events $X = x_i$ and $Y = y_j$ are independent for any x_i, y_j, i. e.

$$P(\{m \in M : X(m) = x_i \text{ and } Y(m) = y_j\}) = P(X = x_i)P(Y = y_i).$$

24. Are the random variables X_1 and X_2 independent in the 1×2 grid (see the definition before Pr. 19)?

25. Prove that if the random variables X and Y are independent, then the mathematical expectation of their product is the product of their mathematical expectations: $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

26. (Wilson's area law) Let $c = 2$, M and N are arbitrary. Compute $\mathrm{E}(W')$ and the energy $E_{M,2}(N)$. For which M_1, N_1, M_2, N_2 the $M_1 \times N_1$ grid has smaller $\mathrm{E}(W')$ than the $M_2 \times N_2$ grid?

27. (Quarks confinement in 2-dimensional space.) Is the amount of energy required to move a quark and an antiquark far away from each other finite or infinite in 2-dimentional space?

28. Investigate 3-dimensional grid $N \times N \times N$ and 4-dimensional grid $N \times N \times N \times N$ experimentally by a numeric simulation for various $c \in [2; 3]$. For them, $S'[U]$ is still defined as the number of 1×1 squares with an odd number of signs "−", and W' is defined as the product of signs along the boundary of one of the 2-dimensional faces. Is the amount of energy required to move a quark and an antiquark far away from each other in 3- and 4-dimentional space finite or infinite?

29. Prove that: a) the ratio of the energies $E_{M,c}(N)/E(N)$ does not depend on N
b) for $c \to \infty$ the energy $E_{M,c}(N)$ tends to the minimal possible value of $S'[U]$, i.e., to 0.

Geometrical view on gauge theory

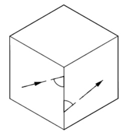
Let us look at our model from the new side (this text is informal and is not used further.)

The charges of the particles can be positive or negative. It is a convention which one to consider positive: is just a choice of direction on the real axis.

And what happens if the choice of the "axis direction" at each point of the space is done independently? For example, in each city of the $M \times N$ grid, choose the direction in its own way. Then for each road you need to specify how the direction of the axis changes as you move from the city to its neighbor. One can imagine a situation, when bypassing a 1×1 square, the direction of the axis is changed to the opposite one (as the direction of the perpendicular to the Möbius strip when it is bypassed.) Geometrical view on gauge theory is that such "defective" squares *are* the magnetic field, and they carry energy. Therefore we computed energy through the number of "defective" squares in our models.

Of course, this construction can not be understood literally: when bypassing, a charge sign, of course, is not changed. The "axis direction" is something different from the charge sign.

30. (Gauss–Bonnet theorem) Consider: a) a cube; b) a regular tetrahedron; c)* a convex polytope. Two vectors lying in adjacent faces are called *parallel*, if they form equal "vertical" angles with the common side of the faces (i.e., they become parallel and looking in the same direction, if one "unfolds" the two faces around the common side; see the figure). Let f_1, f_2, ..., f_k be all faces around a vertex v in the natural order. Start with an arbitrary vector $\vec{e}_1 \subset f_1$ and take a vector $\vec{e}_2 \subset f_2$ parallel to $\vec{e}_1$, then a vector $\vec{e}_3 \subset f_3$ parallel to $\vec{e}_2$,..., and, finally, a vector $\vec{e}_{k+1} \subset f_1$ parallel to $\vec{e}_k$. Denote by ϕ_v the oriented angle between $\vec{e}_{k+1}$ and $\vec{e}_1$. Find the sum of the angles ϕ_v over all vertices v.



Quark color is not a number, but a vector for which three directions are possible. This is why *permutations* of three colors are put on the roads in the theory of strong (i.e., color) interaction.

Permutations

A *permutation* of a set is an arrangement of the elements of the set in some order. More strictly, a *permutation* of the set is a one-to-one correspondence of this set onto itself (i.e. a bijection). The permutation of the set $\{a_1, a_2, \dots, a_n\}$, that to an element a_k assigns the element $f(a_k)$, is written as follows

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ f(a_1) & f(a_2) & \dots & f(a_n) \end{pmatrix};$$

usually $a_k = k$ for all $k = 1, \dots, n$. The *composition* of permutations f and g is the permutation $f \circ g$ defined by the formula $(f \circ g)(x) := f(g(x))$.

31. Find the compositions: (a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

32. Is it true that $x \circ y = y \circ x$ for any pair of permutations x and y?

Toy model for non-Abelian gauge theory

Fix a number $c > 1$. Denote by S_3 a set of all permutations of the set $\{1, 2, 3\}$. A *fixed point* of the permutation f is such $x \in \{1, 2, 3\}$ that $f(x) = x$. The *trace* $\mathrm{Tr}(f)$ of a permutation f is the number of its fixed points minus 1. For example, $\mathrm{Tr}\left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right) = 0$.

To each road AB of the $M \times N$ grid randomly assign a permutation $U(AB) \in S_3$ so that $U(AB) \circ U(BA)$ is the identical permutation, and the probability of an assignment U is equal to

$$P[U] := \frac{c^{-S''[U]}}{\sum\limits_{\text{all the assignments } V} c^{-S''[V]}},$$

where

$$S''[U] := \sum_{\substack{\text{all } 1 \times 1 \text{ squares } ABCD \\ \text{bypassed counterclockwise}}} (2 - \mathrm{Tr}(U(AB) \circ U(BC) \circ U(CD) \circ U(DA))).$$

(I.e. the fewer elements remain fixed when bypassing the square, the greater contribution it makes to $S''[U]$.) Let W'' be the trace of the product of all permutations on the boundary sides bypassed counterclockwise and starting from the lower-left corner. The value $-\frac{1}{M} \log_2 \mathrm{E}(W'')$ is called the *energy of strong interaction of a quark and an antiquark at distance N.*

33. * a) Invent the notion of an *expectation* of a random permutation from S_3 having as many properties of the expectation of a random variable as possible.

Hint. We lack the operation of *adding* permutations. It appears by itself, if we recall that $1, 2, 3$ are quark colors, colors are vectors, and permutation is a mapping.

b) Compute the expectation of W'' for $c = 2$. Prove that there are $a, b, B \in \mathbb{R}$ such that for all M, N we have $ba^{MN} \leqslant \mathrm{E}(W'') \leqslant Ba^{MN}$.

34. (The "essence" of the Millenium problem) Prove the same inequality for the 4-dimensional $N \times N \times N \times N$ grid and a rectangle $M \times N$ on one of its 2-dimensional faces.

Additional problems

These hard problems are given to those who have solved most of the problems above.

35. Solve the analogue of Problem 33.b, if the definition of permutation trace is changed as follows:

$$\mathrm{Tr}'(x) = \begin{cases} +3, & \text{if } x \text{ is identical, i.e. } x = \left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right); \\ 0, & \text{if } x \text{ is a cycle of length 3, i.e. } x = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right) \text{ or } x = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right); \\ -1, & \text{if } x \text{ is a transposition, i.e. } x = \left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right), x = \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right), \text{ or } x = \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right), \end{cases}$$

and the expression $2 - \mathrm{Tr}(\dots)$ is replaced by $3 - \mathrm{Tr}'(\dots)$ in the formula for $S''[U]$.

36. (Strong coupling expansion) Consider the solid model on a 3-dimensional grid $N \times N \times N$; see Problem 28. Find an asymptotic form for $\mathrm{E}(W')$ as $c \to \infty$, i.e., a function $f_N(c)$ such that $\lim\limits_{c\to\infty} \frac{\mathrm{E}(W')}{f_N(c)} = 1$ for fixed N.

Underwater riffs

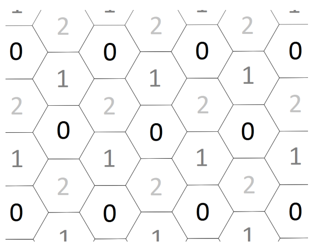
We hope that at least some of our readers have become interested in elementary particles and want to learn more about them. As an epilogue, let us give a few warnings to such readers.

In popular science, theory of elementary particles is usually oversimplified. This sequence of problems is not an exception. The toy models introduced here are very rough and should be considered with a grain of salt. Simplicity is their only advantage; if taken too seriously, the models could even give a wrong physical intuition. Only the model called "solid model for quantum gauge theory" is a respectful one truly considered in physical literature. Real understanding of particles theory requires excellent knowledge of both physics and mathematics.

The problem called "the essence of the Millenium Problem" is *not* equivalent to the Millenium Problem in Yang–Mills theory at all. It is *not* even a particular case. It is rather the most essential — in our very subjective opinion — part of the problem freed from technical details. Although we came to the problem during the discussion of quarks confinement, it is closer to another phenomenon: *short range of nuclear forces*.

We should also remark that nowadays there are almost no mathematical results in lattice gauge theory; what we have is usually just a numeric simulation. Finally, there are "*theories of New Physics*" which are developed without any objective truth criterion: such theories are supported by neither experimental nor mathematical proofs (and some of them have experimental disproofs).

Hints, solutions, answers



1. *Answer:* any integer multiple of 3.
Hint. Let us write the remainders 0, 1 and 2 (when dividing by three) into cells as in the figure below. Note that a positive move from any cell adds 1 to the remainder on which the king stands, and a negative move subtracts 1.
2. *Answer:*

proton	p	uud	$+1$	antiproton	$\bar{p}$	$\bar{u}\bar{u}\bar{d}$	-1
neutron	n	udd	0	antineutron	$\bar{n}$	$\bar{u}\bar{d}\bar{d}$	0

3. a) *Answer*: $(q, p) = (1, 1)$, $(3, 0)$ or $(0, 3)$.

Hint. This problem is solved by a brute-force exhaustion, let us give an example of reasoning for one case. Why q and p can not be equal to 2 and 0 respectively: if the sum of two vectors is 0, then they are opposite, but two quarks can not have opposite colors.

b) Let $p \leqslant q$. Note that the sum of the charges of an arbitrary quark and an antiquark is an integer. Hence, we can consider only the sum of the charges of $q - p$ quarks. Also note that the sum of the charges of any three quarks is an integer. It follows from Problem 1 that for a particle with zero color $q - p$ is a multiple of 3, that is, $q - p$ quarks can be divided into triples.
4. *Answer*: $x = 0$. *Hint*:

$$S[U] = \log_2^2(2 \cdot 1 \cdot \frac{1}{2^x} \cdot 1) + \log_2^2(2^x \cdot 1 \cdot 2 \cdot 1) = (1-x)^2 + (1+x)^2 = 2 + 2x^2.$$

The minimum is reached at $x = 0$.

6. *Answer.* In the table below, in the third and fourth lines in the first three columns there are values for all non-boundary roads (from top down). It is assumed that all these roads are directed from left to right. In the same lines in the last column it is considered that all non-boundary roads are directed from the center to the borders, and by x we denote an arbitrary real number (that is, there is a solution for each x). If for all roads (or squares) the value is the same, then only one number is placed in the corresponding cell.

Grid	1×2	1×3	$1\times N$	2×2
The value W	4	4	4	16
Minimal $S[U]$	2	$\frac{4}{3}$	$\frac{4}{N}$	4
Optimal rates	1	$2^{-1/3}$, $2^{1/3}$	$2^{-1+\frac{2}{N}}, 2^{-1+\frac{4}{N}}, \dots 2^{-1+\frac{2N-2}{N}}$	2^x
Values $x(AB)$	0	$-\frac{1}{3}$, $\frac{1}{3}$	$-1+\frac{2}{N}, -1+\frac{4}{N}, \dots -1+\frac{2N-2}{N}$	x
Values $x(ABCD)$	1	$\frac{2}{3}$	$\frac{2}{N}$	1

Hint. In the solution of this problem, as well as some subsequent ones, it is very useful *the inequality of arithmetic and quadratic means (AM-QM inequality)*, which is formulated as follows:

for all real numbers $x_1, x_2, \dots x_n$ the inequality holds:

$$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geqslant \frac{x_1 + x_2 + \ldots + x_n}{n},$$

and equality is achieved if and only if $x_1 = x_2 = \ldots = x_n \geqslant 0$.

For the $1 \times N$ grid one should take the values of $x_1, x_2, \dots x_n$ equal to $x(ABCD)$ for all squares $ABCD$ (enumerated from top down).

For the 2×2 grid denote by 2^x, 2^y, 2^z and 2^t the exchange rates for non-boundary roads directed from the center to the boundary and going counterclockwise. Then

$$S[U] = (x+1-y)^2 + (-z+y+1)^2 + (1-t+z)^2 + (1-x+t)^2 =$$

$$= 2x^2 + 2y^2 + 2z^2 + 2t^2 + 4 - 2xy - 2yz - 2zt - 2tx =$$

$$= (x-y)^2 + (y-z)^2 + (z-t)^2 + (t-x)^2 + 4 \geqslant 4.$$

Remark. There are other solutions, some of them rely on the *existence* of optimal rates. In this case, the latter must be proved, and this is rather difficult, see [SSU, The principle of achieving the smallest value on p.30].

7a. (*E. Pavlov*) *Answer*: $S[U] = 0$ can be achieved if and only if $W = 1$.

Indeed, $S[U] = 0$ if and only if for each 1×1 square $ABCD$ we have $U(ABCD) = 1$. Consider the product

$$\prod = \prod_{\substack{\text{all } 1\times 1 \\ \text{squares } ABCD}} U(ABCD).$$

In this product each non-boundary road appears twice, once as $U(AB)$ and once as $U(BA)$, cancelling each other. The boundary roads appear once with a counterclockwise orientation. This means that $\prod = W$. Since every term in $\prod$ is just 1 it follows that $W = \prod = 1$ is a necessary condition.

Now assume that $W = 1$. Let us show that there exist rates $U(AB)$ such that $S[U] = 0$. First, we denote by $a_0, a_1, \cdots, a_{2M+2N-1}$ the vertices on the boundary, starting from the bottom left vertex. Next, we introduce the function $P(A)$ on the vertices of the grid by:

$$P(A) = \begin{cases} 1, & \text{if } A \text{ is not on the boundary or the bottom left vertex.} \\ \prod_{k=0}^{n-1} U(a_k a_{k-1}), & \text{if } A \text{ is the vertex } a_n \text{ on the boundary.} \end{cases}$$

Finally, we extend $U(AB)$ to the non-boundary roads by the formula $U(AB) = \frac{P(B)}{P(A)}$. For each square $ABCD$, we have

$$\log_2^2 \left(U(AB)U(BC)U(CD)U(DA)\right) = \log_2^2 \left(\frac{P(B)}{P(A)}\frac{P(C)}{P(B)}\frac{P(D)}{P(C)}\frac{P(A)}{P(D)}\right) = \log_2^2(1) = 0.$$

This means that $S[U] = 0$, q.e.d.

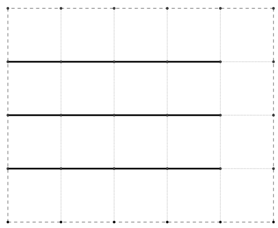
7b*. It suffices to construct a function $P(A)$ on the set of cities such that $U(AB) = \frac{P(B)}{P(A)}$ for each road AB. For a city A, set $P(A)$ to be the product of the rates for the Γ-shaped route starting from the bottom-left city and ending at A. Then the condition $U(ABCD) = 1$ for each 1×1 square $ABCD$ implies the condition $U(AB) = \frac{P(B)}{P(A)}$.

7c. For $M, N > 1$ the rates are not unique. For an interior city, one can change the units of measurements, e.g., exchange dozens of apples instead of single ones. Such *gauge transformation* multiplies the rates for all the roads starting from the city by the same value but preserves $S[U]$.
7d. *Answer*: $2^{\sqrt{M\cdot N\cdot S_{min}}} = W$, where S_{min} is the minimal value of $S[U]$.

Hint. By AM-QM inequality we have:

$$S[U] = \sum_{\substack{\text{all } 1\times 1\\ \text{squares } ABCD}} x(ABCD)^2 \geqslant \frac{1}{MN}\left(\sum_{\substack{\text{all } 1\times 1\\ \text{squares } ABCD}} x(ABCD)\right)^2 = \frac{\log_2^2 W}{MN}$$

It remains to prove that it is always possible to establish such exchange rates in the kingdom so that equality is achieved, that is, the values of $x(ABCD)$ are equal to $\frac{\log_2 W}{MN}$ for all 1×1 squares. For a rectangle $1\times N$ one can successively determine the rates on the undefined roads, bypassing them from the top down. These rates are uniquely determined, and for the last square $x(ABCD)$ is equal to $\frac{\log_2 W}{MN}$, because the sum of these quantities is fixed for all squares. For the rectangle $M\times N$, one can establish an exchange rate equal to 1 on the roads drawn in bold lines (see the figure). And then establish the rates on the remaining roads as written above.



7e. *Answer*: 8×8. This follows from the formula in the problem 7d.
8. *Answer*: $-MNj^2/2$.

Hint. Denote $w = \log_2 W$. For fixed w, the minimum of $S[U]$ equals $\frac{w^2}{MN}$ by Problem 7.d. It remains to find the minimum of the quadratic polynomial $\frac{w^2}{2MN} - jw$.
9a. *Answer*: it tends to increase the area.

The first step in solving this problem (and many others) is *mathematical formalization*. Problem a) is formalized as follows: *Consider the minimal value of* $\frac{1}{2}S[x] - j\log_2 W$ *on the* $M\times N$ *grid as a function of* MN *with fixed* j. *Is this function increasing or decreasing?* After such formalization, the solution follows from Problem 8.

The work of any electric motor and generator is based on this fact.
9b. *Answer*: opposite currents repulse.

Mathematical formalization: *consider the minimal value of* $\frac{1}{2}S[x] - j\log_2 W$ *on the* $M\times N$ *grid as a function of* N *with fixed* M *and* j. *Is this function increasing or decreasing?*
9c*. *Answer*: currents with the same directions attract.

Mathematical formalization: *consider the minimal value of* $\frac{1}{2}S[x] - j\log_2 W$ *on the* $M\times(L-N)$ *grid as a function of* N *with fixed* L, M, *and* j. *Is this function increasing or decreasing?*
10b. *Answer*: $\frac{1}{2}$, 1, $\frac{3}{2}$, $\frac{N}{2}$.

Take an arbitrary square of the $1\times N$ grid and consider a random variable equal to 0 if there is an even number of signs "$-$" on its sides, and 1, if odd. By a not very long brute-force search it is easy to check that 0 and 1 are equally probable, hence, the mathematical expectation of this variable is $\frac{1}{2}$. Note that the value $S'[U]$ is equal to the sum of such random variables over all 1×1 squares. Since mathematical expectation of sum is equal to sum of mathematical expectations, we obtain $E(N) = \frac{N}{2}$.
11. *Answer*: no.

Hint. In this model the energy $E(N) = \frac{N}{2}$ tends to infinity as $N\to\infty$.
12. *Answers*: a) $\frac{1}{2}$; b) $\frac{1}{13}$; c) $\frac{3}{13}$; d) $\frac{1}{52}$; e) $\frac{15}{52}$.
13. *Answers*: a) $\frac{1}{8}$; b) $\frac{3}{8}$.
14a. *Answer*: $P(A\cup B) = P(A) + P(B)$.

Let A and B be subsets of the set M. Since A and B do not intersect, $|A\cup B| = |A| + |B|$. Hence, by definition of probability we get

$$P(A\cup B) = \frac{|A\cup B|}{|M|} = \frac{|A|+|B|}{|M|} = \frac{|A|}{|M|} + \frac{|B|}{|M|} = P(A) + P(B).$$

14b. *Answer*: $P(A\cup B) = P(A) + P(B) - P(A\cap B)$.
14c. *Answer*: $P_{M\times N}(A\times B) = P_M(A)P_N(B)$.
15. *Answers*: a) $\frac{1}{2}$; b) $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

We give an example of the reasoning for b). We divide into pairs all the arrangements of signs in the following way: two arrangements are in the pair if and only if they differ *only* by the sign on the upper side. Note that replacing the sign on the upper side changes the parity of the number of "$-$" signs on the sides of the upper square. Hence, the number of arrangements with $S'[U] = 1$ equals to the number of all arrangements with values $S'[U] = 0$ and $S'[U] = 2$ taken together. That is a half of the total number of arrangements. Similarly, we divide into pairs all the arrangements that give $S'[U]=0$ and $S'[U]=2$, and differ only in the sign on the central side. The same as above argument gives us that the number of assignments with $S'[U]=0$ equals to the number of assignments with $S'[U]=2$.

16. *Answers:* a) yes; b) no; c) no.

17. *Answers:* $\{(0,1/4),(1,1/2),(2,1/4)\}$. This is actually Problem 15.

18. *Answer:* 1.

19. *Answer:* $S'[U]=(2-X_1-X_2)/2$, $W'=X_1X_2$, $\mathrm{E}(X_1)=\mathrm{E}(X_2)=\mathrm{E}(W')=0$.

20. *Answers:* c) μ; d) yes; e) no.

Counterexample to e): $\mathrm{E}(X_1^2)=\mathrm{E}(1)=1\neq 1/4=\mathrm{E}(X_1)\mathrm{E}(X_1)$.

21. a) *Answer:* The probability of an assignment with an even number of signs "-" is $\frac{c}{8(c+1)}$, with an odd — $\frac{1}{8(c+1)}$; $\mathrm{E}(W')=\frac{c-1}{c+1}$, $E_{1,2}(1)=\log_2 3$.

b) *Answer:* The probability of the assignment without signs "-" is $\frac{c}{c+1}$, with a sign "-" — $\frac{1}{c+1}$; $\mathrm{E}(W')$ and $E_{1,2}(1)$ are the same as in a).

22. *Answer:* the probability does not change.

23. *Answers:* a) yes; b) no.

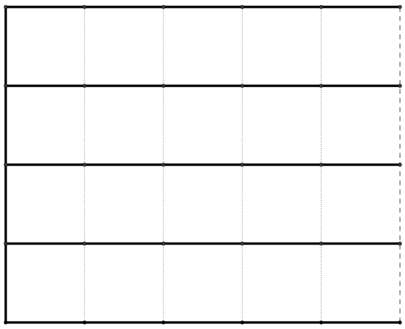
24. *Answer:* yes.

Hint. The following simplification of the solid model is very useful for this and many other problems. Changing the signs of all the roads coming from one particular city is called a *gauge transformation*. By Problem 22 this does not change the probability of an assignment. It changes neither the random variables X_k nor W'. For an arbitrary assignment, by a sequence of gauge transformations one can turn all the signs at the solid segments in the figure into pluses. Thus one may assume that only the assignments such that all the solid segments have signs "+" are allowed in the model. Next, the signs of all the other segments are uniquely determined by the values of X_k at all the squares. Thus our model is equivalent to the following *Bernoulli process*:

Let the probability of a collection of numbers $X_1,\dots,X_{MN}\in\{+1,-1\}$ be

$$P(X_1,\dots,X_{MN})=\frac{c^{(MN-X_1-\dots-X_{MN})/2}}{\sum\limits_{\text{all possible collections }(Y_1,\dots,Y_{MN})} c^{(MN-Y_1-\dots-Y_{MN})/2}}.$$

Then the energy $E_{M,c}(N)=-\frac{1}{M}\log_2\mathrm{E}(X_1\cdot\dots\cdot X_{MN})$.



26. *Answer:* $\mathrm{E}(W)=\frac{1}{3^{MN}}$, $E_{M,2}(N)=N\log_2 3$.

Hint. Use the simplified model from the solution of Problem 24. Show that $X_1,\dots,X_{MN}$ are independent, hence $\mathrm{E}(X_1\cdot\dots\cdot X_{MN})=\mathrm{E}(X_1)\cdot\dots\cdot\mathrm{E}(X_{MN})$. (Do not forget to define *independence* for more than 2 random variables.) Show that each $\mathrm{E}(X_k)$ does not depend on the size of the grid, hence $\mathrm{E}(X_k)=1/3$ by Problem 21.

27. *Answer:* infinite. This follows from Problem 26.

28. For the $4D$-grid, a strong phase transition at $c=1+\sqrt{2}$ should be visible [1, (16.37) and Figure 9.1].

30. a)–c) *Answer*: 4π. *Hint*: compute the sum of the angles of all the faces in two ways.

32. *Answer*: no.

33. a) To each permutation $f\colon\{\overrightarrow{OR},\overrightarrow{OG},\overrightarrow{OB}\}\to\{\overrightarrow{OR},\overrightarrow{OG},\overrightarrow{OB}\}$ assign the 2×2 matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, where the real numbers a,b,c,d are given by the equations

$$f(\overrightarrow{OR})=a\,\overrightarrow{OR}+c\,\overrightarrow{OG};$$
$$f(\overrightarrow{OG})=b\,\overrightarrow{OR}+d\,\overrightarrow{OG}.$$

Then the *expectation* of a random permutation is defined literally by the same formula as the expectation of a random variable, but addition and multiplication are understood as operations over matrices. Beware that such expectation is a 2×2 matrix rather than a permutation.

b) *Answer:* $E(W'') = 2\left(\frac{7}{16}\right)^{MN}$. *Hint.* Argue analogously to the solution of Problem 26. First compute $E(U(ABCD)) = \begin{pmatrix} 7/16 & 0 \\ 0 & 7/16 \end{pmatrix}$. Then show that random permutations $U(ABCD)$ are independent for distinct 1×1 squares $ABCD$, hence $E(X_1 \circ \dots \circ X_{MN}) = E(X_1)\dots E(X_{MN}) = \begin{pmatrix} (7/16)^{MN} & 0 \\ 0 & (7/16)^{MN} \end{pmatrix}$. Finally, define the *trace* of a 2×2 matrix by $\mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := a + d$ and show that $E(\mathrm{Tr}(W'')) = \mathrm{Tr}(E(W''))$.

35. *Answer:* $E(W'') = \left(\frac{17}{23}\right)^{MN} + 2\cdot\left(\frac{14}{23}\right)^{MN}$. *Hint.* Argue analogously to the solution of Problem 33.b.

36. *Semi-answer:* $E(W') \simeq \mathrm{const}/c^{N^2}$.

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