## 13TH HILBERT PROBLEM ABOUT SUPERPOSITIONS OF FUNCTIONS [1]

### Problems before the Semifinal

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### What is this collection of problems about

This project is devoted to several classical results and methods in pure mathematics. They are also interesting from the point of view of computer science (related to combinatorial geometry and coding theory).

Suppose there are several functions. Then some of them may be used as arguments of others. This operation is called *superposition*. For example,

- function $f:\mathbb{R}\to\mathbb{R}$, $f(x,y)=x^2y+y^2$ is a superposition of $x+y$ and $xy$;
- function $f:\mathbb{Z}_2^2\to\mathbb{Z}_2$, $f(x,y)=x\oplus y=x$ XOR $y$ is a superposition of $\overline{x}$, $x\vee y$ and $x\wedge y$;
- function $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z)=g\Big(g\big(\sin x+y, g(y,x^2,z),x\big),x,x\Big)$ is a superposition of $g(x,y,z), x+y, \sin x, x^2$.

The explicit definition is provided at the beginning of §1. Superpositions are important objects in analysis, topology, and computer science. General problem is the following: when given function of several variables may be represented as a superposition of functions of less number of variables? The answer depends on considered class of functions (see [ZSS, п. 21.5 'Superpositions of Boolean functions'], [Ar58]).

We are going to demonstrate most important ideas of solution of the general problem for continuous functions, i.e. the proof of A. N. Kolmogorov's Theorem 1.11 (this is a solution of 13th D. Hilbert's Problem). These ideas will be demonstrated on the «olympiade» examples. These examples are simplest particular cases which are free from technical details. As a result this collection of problems is accessible for beginners although it contains beautiful and complicate results.

No specific knowledge is required to solve these problems. All necessary definitions are presented here. But you would need some cleverness and mathematical culture (which will be imporoved as a result of solving these problems). In particular, this collection of problems is analytic. However only minimal knowledge of mathematical analysis is required for many problems. Main ideas for solution are demonstrated on discrete versions of problem. As a result the solving of these problems assists to develop analytical experience and intuition.

We are going to propose several beautiful problems for further research. Some well-known exapmles from «continuous» mathematics (Weierstrass function, Peano curve) are useful in physics and computer science. We believe that similar but less known ideas of Kolmogorov's Theorem will also be useful.

### Conventions

If a problem is a statement then a proof of this statement is required in this problem. Basic problems are marked with a circle (like 5°). We do not recommend to try other problems of the same section before you solve basic problems. If the problem is marked with a star (like 5*) then it is more complicated. You can postpone its solution until solving of other problems.

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for every solution which has been written down and marked with either '+' or '+.'.

For every solution which has been written down and marked with either '+' or '+.' a student (or a group of students) get a "bean". The jury may also award extra bean for beautiful solutions, solutions of hard problems, or (some) solutions typeset in TEX. The jury has infinitely many bean. One may submit a solution in the oral form, but one loses a bean with each 5 attempts (successful or not).

If you are stuck on a certain problem we suggest to try looking at the next ones. They may turn out to be helpful. We suggest to all the students working on the project to *consult* the jury on any questions on the project. Students who brilliantly work on the project will get several *extra problems.*

## 1 Definition and examples of superposition

**1.1.** An arrangement of numbers on the chessboard is called *basic* if there are numbers $\varphi_1, \dots, \varphi_8, \psi_1, \dots, \psi_8$ such that number at the square $(i, j)$ is equal to $\varphi_i + \psi_j$ for each square on the chessboard.

(a) Is any arrangement basic?

(b) Suppose the arrangement is basic. Then, for any squares $A, B, C, D$ such that $ABCD$ is a rectangle and whose sides are parallel to board sides, the sum of numbers in $A$ and in $C$ is equal to the sum of numbers in $B$ and in $D$.

(c) Suppose the arrangement is basic. Then, for any closed route of a rook on board with sequential turns at squares $A_1, \dots, A_{2n}$, the sum of numbers at squares $A_1, A_3, \dots, A_{2n-1}$ is equal to the sum of numbers at squares $A_2, A_4 \dots, A_{2n}$.

(d) Is the converse of the statement (b) true?

**1.2.** (a) Is it true that for any arrangement of numbers on the chessboard there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that the number at the square $(i, j)$ is equal to $h(i + \sqrt{2}j)$ for each square on the chessboard?

(b) Is it true that for any arrangement of numbers on the chessboard there are integers $\varphi_1, \dots, \varphi_8, \psi_1, \dots, \psi_8$ and a function $h : \mathbb{R} \to \mathbb{R}$ such that the number at the square $(i, j)$ is equal to $h(\varphi_i + \psi_j)$ for each square on the chessboard?

(c) Are there integers $\varphi_1, \dots, \varphi_8, \psi_1, \dots, \psi_8$ such that for any arrangement of numbers on the chessboard there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that the number at the square $(i, j)$ is equal to $h(\varphi_i + \psi_j)$ for each square on the chessboard?

(d) Are there integers $\varphi_{ik}, \psi_{ik}$, $i, k = 1, \dots, 8$ such that for any arrangement of numbers in the cube $8 \times 8 \times 8$ there is a function $h : \mathbb{R} \to \mathbb{R}$ such that the number in the cell $(i, j, k)$ is equal to $h(\varphi_{ik} + \sqrt{2}\psi_{jk})$ for each cell in the cube?

Let $A$ be $\mathbb{R}$ or $\mathbb{Z}_q$ and $M \subset A$. *A polynomial with coefficients in the set $A$* is an infinite sequence $(a_0, \dots, a_n, \dots)$ of numbers from $A$ such that only finitely many of $a_n$ are nonzero. For any polynomial (i.e. the sequence) there is corresponding function $\overline{P}\colon M \to M$ which is defined as $\overline{P}(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots$ (this sum is finite). The polynomial $P = (a_0, \dots, a_n, \dots)$ is denoted by $P(x) = a_0 + a_1 x + \ldots + a_n x^n$.

For any set $X$ we denote $X^n = \{(x_1, \dots, x_n) \ : \ x_1, \dots, x_n \in X\}$.

**1.3.** Which functions are polynomials (more precising, correspond to some polynomial)?
(a) $\sin x$ on $\mathbb{R}$; (b)* $\sin x$ on $[0, 1]$; (c)° $\sin x$ on $\{0, 1\}$; (d) $\sin x$ on $\{0, \frac{1}{9}, \frac{2}{9}, \dots, \frac{8}{9}, 1\}$.

**1.4.** (a) Give the 'definition' of function (mapping) $f : X \to Y$.

(b) Any function $\mathbb{Z}_q \to \mathbb{Z}_q$ is a polynomial for a prime $q$.

(c) Any function $\mathbb{Z}_q^n \to \mathbb{Z}_q$ is a polynomial for a prime $q$.

**1.5.** A *level line* and a *graph* of function $f : \mathbb{R}^2 \to \mathbb{R}$ are sets

$$f^{-1}(c) := \{(x, y) \in \mathbb{R}^2\colon f(x, y) = c\} \quad \text{и} \quad \{(x, y, f(x, y)) \in \mathbb{R}^3\colon (x, y) \in \mathbb{R}^2\}.$$

correspondingly. Draw level lines and graphs for following functions:
(a) the distance to the point; (b) distance to the line; (c) the sum of distances to two points;
(d)* the product of distances to two points; (e) the quotient of distances to two points;
(f) $f(x,y) = x + y$; (g) $f(x,y) = xy$; (h) $f(x,y) = x/y$.
(In (e,h) function is defined in the *subset* of plain.)

**Definition of superposition.** Suppose $F = \{f_\alpha(x_1, \dots, x_{n_\alpha})\}_{\alpha\in A}$ is the set of functions (not necessarily finite). Then the set $\overline{F}$ of *superpositions* of functions from $F$ is the set of all functions which may be constructed from elements of $F$ and all variables $x_j$ by the sequence of *elementary superpositions*. The operation of *elementary superpositions* is the following:

if we already have a functions $f(x_1, \dots, x_n)$, $g_1(\dots)$, $g_2(\dots)$, $\dots$, $g_n(\dots)$ we can take $f(g_1(\dots), \dots, g_n(\dots))$.

Any variables can be used as arguments of $g_i$. Variables may be coinciding. The range of the internal function $g_i$ which is used as argument must be subset of the domain of external function $f$.

Examples of superpositions are provided at the beginning of the text. The statement 1.4.c means that any function $\mathbb{Z}_q^n \to \mathbb{Z}_q$ is a superposition of the constant 1, the addition modulo $q$, and the multiplication modulo $q$. If the set $F$ contains constants (i.e. numbers, or functions without arguments) and the addition and multiplication of two variables then $\overline{F}$ contains all polynomials $\sum\limits_{k_1,\dots,k_n} a_{k_1,\dots,k_n} x_1^{k_1} \dots x_n^{k_n}$.

**1.6.** Is it true that
(a) $xy$ is a superposition of functions of one variable?
(b) $x^3y + xy^2 \in \overline{\{xy, x+y\}}$? (c) $xy \in \overline{\{x+y\}}$? (d) $xy \in \overline{\{x+y, x/n, x^n\}_{n\in\mathbb{Z}-\{0\}}}$?
(e) $xy$ as a function $(0,+\infty)^2 \to (0,+\infty)$ lies in $\overline{\{x+y, 2^x, \log_2 x\}}$?
(f) any function of one variable lies in $\overline{\{x+y, 2^x, \log_2 x\}}$?
(g) $\sin x \in \overline{\{x+y, xy\}}$? (h) $\sin x \in \overline{\{x+y, xy, 2^x\}}$? (i)* $\sin x \in \overline{\{x+y, xy, 2^x\} \cup \{c\}_{c\in\mathbb{R}}}$?
(j) function $g(x_1, x_2, x_3) = x_1^{\frac{x_2}{x_3}}$ with $x_1 > 1, x_2, x_3 > 0$ lies in $\overline{\{x_1 - x_2, 2^x, \log_2 x\}}$?

**1.7.** $(a)^\circ$ The function of two or more variables is not a superposition of functions of one variable.
(b) If the set $F$ is finite or countable then $\overline{F}$ is no more than countable.

Denote by $F_n$ the set of *all* functions $[0,1]^n \to [0,1]$, i. e. functions of $n$ variables.

**1.8.** (a) There exists an injection $\alpha : [0,1]^2 \to [0,1]$ (i.e. mapping such that $\alpha(x) \neq \alpha(y)$ for any $x \neq y$).
(b) $F_2 \subset \overline{F_1 \cup \{\alpha\}}$. (c) $F_3 \subset \overline{F_2}$.
(d) $F_n \subset \overline{F_{n-1}}$ for any $n \geqslant 3$. (e) $F_n \subset \overline{F_1 \cup \{\alpha\}}$ for any integer $n$.

**1.9.** (a) Are there continuous functions $\varphi_1, \varphi_2, \psi_1, \psi_2 : [0,1] \to [-1,1]$ and $h_1, h_2 : [-2,2] \to \mathbb{R}$ such that for every $x, y \in [0,1]$ we have $xy = h_1(\varphi_1(x) + \psi_1(y)) + h_2(\varphi_2(x) + \psi_2(y))$?
(b) Are there continuous functions $\varphi, \psi : [0,1] \to \mathbb{R}$ and $h : [0,2] \to \mathbb{R}$ such that for every $x, y \in [0,1]$ we have $(x+1)(y+1) = h(\varphi(x) + \psi(y))$?

**1.10.** (a) $F_2 \subset \overline{F_1 \cup \{x+y\}}$.
(b) Let $f : [0,1]^2 \to [0,1]$ be an arbitrary function of two variables. Then there are functions $\varphi, h : [0,1] \to [0,1]$ such that for every $x, y \in [0,1]$ we have $f(x,y) = h(\varphi(x) + 0.1\varphi(y))$.
(c) $F_n \subset \overline{F_1 \cup \{x+y\}}$ for any integer $n$.
(d) Let $n$ be any natural and let $f : [0,1]^n \to [0,1]$ be an arbitrary function of $n$ variables. Then there are functions $\varphi, h : [0,1] \to [0,1]$ such that for any $x_1, \dots, x_n \in [0,1]$ we have

$$f(x_1, \dots, x_n) = h\left(\varphi(x_1) + 10^{-1}\varphi(x_2) + \ldots + 10^{1-n}\varphi(x_n)\right).$$

So, any function is a superposition of some functions of one variable and the addition. However it is more interesting to consider the set of generators which contains only *continuous* (see definition in §3) or *infinitely differentiable* functions.

**1.11. Kolmogorov's Theorem.** (a) *Any continuous function $f : [0,1]^n \to \mathbb{R}$ may be represented as a superposition of continuous functions of one variable and the addition.*

(b) *Let $f : [0,1]^2 \to \mathbb{R}$ be an arbitrary continuous function. Then there are continuous functions $\varphi_1, \varphi_2, \dots, \varphi_5 : [0,1] \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in [0,1]$ we have*

$$f(x,y) = h\left(\varphi_1(x) + \sqrt{2}\varphi_1(y)\right) + \ldots + h\left(\varphi_5(x) + \sqrt{2}\varphi_5(y)\right).$$

(c) *Let $p_1, \dots, p_n$ be different prime numbers. Then for any $n$ and continuous function $f : [0,1]^n \to \mathbb{R}$ there are continuous functions $\varphi_1, \varphi_2, \dots, \varphi_{2n+1} : [0,1] \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in [0,1]$ we have*

$$f(x_1, \dots, x_n) = \sum_{k=1}^{2n+1} h\left(\sqrt{p_1}\varphi_k(x_1) + \ldots + \sqrt{p_n}\varphi_k(x_n)\right).$$

Hints for (a) are presented after the Semifinal.

## 2 Rough estimations

First problems in this section are the simplest way to understand the concept of continuous function. But they are interesting not only because of this. Similar problems about specific (but may be rough) estimations are occuring very often in olympiads and in the applied and theoretical mathematics.

In the solution of these problems you may not use functions $\sqrt[n]{x}$, $a^x$, $\log_a x$, $\arcsin x$ etc. before rigorous definitions of these functions. It because to define these functions rigorously (e.g. to prove that there is $x$ such that $x^2 = 2$), in fact, you should solve the corresponding problem. However there is one exception from this rule. If some function is used in the condition of the problem then you may use it in the solution.

In this collection of problems the rigorous theory of real numbers is not necessary. You may use algebraic properties of real numbers without proof. In particular, you may use properties of inequalities. You may also use:

*Archimede's principle*: for any real number $x$ there is an integer number $n$ such that $n$ greater than $x$.

*Principle of nested intervals*: Let $I_n$ be a system of nested intervals on the line and a length of $I_n$ tends to 0. Then $\bigcap I_n \neq \varnothing$.

All these principles can be «proven» by decimal notation.

**2.1.** Find $N$ such that for every $n > N$ the inequality $a_n > 10^9$ is satisfied for $a_n =$
(a) $\sqrt{n}$;    (b) $n^2 - 3n + 5$;    (c) $1,02^n$.

**2.2. Bernoulli's Inequality.** $(1+x)^k \geqslant 1 + kx$ for every $x \geqslant -1$ and every integer $k \geqslant 1$.

**2.3.** Find a pair of $a$ and $N$ such that for every $n > N$ the inequality $|a_n - a| < 10^{-8}$ is satisfied if $a_n =$
(a) $\dfrac{n^2 - n + 28}{n - 2n^2}$;    (b) $\sqrt{5 + \dfrac{2}{n}}$;    (c) $0,99^n$;    (d) $\sqrt[n]{2}$;    (e)* $\dfrac{n^9}{2^n}$.

**2.4.** Find a pair of $a$ and $\delta > 0$ such that for every $x \in (-\delta, \delta)$ the inequality $|f(x) - a| < 3 \cdot 10^{-9}$ is satisfied if $f(x) =$
(a) $(x-3)^3$;    (b) $3^{x-3}$;    (c) $\sin x$;    (d) $\dfrac{\sqrt{1+x^5}}{\cos x - 2}$;
(e) the root of equation $t^3 - tx + 1$ which lies in $[-2, 0]$.

## 3 Continuous functions

Let $K = [0,1]$ or $K = [0,1]^2$. A function $f : K \to \mathbb{R}$ is called **continuous** if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $x, y \in K$ with condition $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Here $|x - y|$ is the euclidean distance. (Be careful because for other $K$ the definition may be different!)

**3.1.** (a)-(e) Which functions from the problem 2.4 are continuous on $[0,1]$?

Note. The continuity of the function $f(x) = x^n$ for any integer $n > 0$ can be proven similarly to problems 2.4.a and 3.1.a. This fact and the intermediate value theorem imply that for any $a > 0$ there exists a number $x$ such that $x^n = a$. This statement allows to define the function $\sqrt[n]{x}$. Inverses of other functions can be defined similarly.

**3.2.** Which functions are continuous in $[0,1]^2$?
(a) $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$; (b) $f(x_1, x_2) = \lfloor x_1 + x_2 \rfloor$.

**3.3.** (a) Any continuous function $f\colon [0,1] \to \mathbb{R}$ is bounded (i.e. there exists a constant $M$ such that $|f(x)| < M$ for every $x \in [0,1]$).

(b) Any continuous function $f\colon [0,1] \to \mathbb{R}$ reach its maximal and minimal values.

**3.4.** Let $f$ and $g$ be continuous functions. Is it true that
(a) its sum; (b) its product; (c) its superposition;
(d) the superposition $f(g(x,y),z)$; (e)* arbitrary superposition
are continuous?

We consider funstions $[0,1] \to [0,1]$ (for (a)-(c)), $[0,1]^2 \to [0,1]$ (for (d)), $[0,1]^n \to [0,1]$ (for (e)).

**3.5.** Is it true that the function which is continuous by any its variable is continuous? In other words, is there a discontinuous function $f\colon [0,1]^2 \to \mathbb{R}$ such that each of its section (i.e. functions $f_y\colon [0,1] \to \mathbb{R}$ and $f_x\colon [0,1] \to \mathbb{R}$ which defined as $f_y(x) := f(x,y)$ и $f_x(y) := f(x,y)$) is continuous?

If a counterexample is a solution of some problem then it does not used further. But contreaxmples are necessary for understanding the context of proof. These counterexamples show you what properties can not be used.

## 4 Uniform limits

Let us recall that $\lim\limits_{n\to\infty} a_n = a$ if for every number $\varepsilon > 0$ there exists an integer $N > 0$ such that for all $n > N$ we have $|a_n - a| < \varepsilon$.

**4.1.** (a) Find $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} x^k$ for each $x \in (0,1)$. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} x^k$.

(b) Is there $N > 0$ such that for every $x \in (0,1)$ and $n > N$ we have $|\frac{1}{1-x} - \sum\limits_{k=0}^{n} x^k| < 0.01$?

**4.2.** Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions. Is it true that following functions are continuous:

(a) The *pointwise limit* of $f_n$? This is a function $f(x) := \lim\limits_{n\to\infty} f_n(x)$ which is defined if all these limits are exist.

(b) The **uniform limit** of $f_n$? This is a function $f : [0,1] \to \mathbb{R}$ such that for every number $\varepsilon > 0$ there exists an integer $N > 0$ such that for all $n > N$ and $x \in [0,1]$ we have $|f_n(x) - f(x)| < \varepsilon$.

**4.3.** (a) Construct a continuous surjective function $f : [0,1] \to [0,1]$ which is a constant outside of an interval of length 0.01.

(b) Construct an infinite sequence of continuous surjective functions $f_n : [0,1] \to [0,1]$ such that

• $|f_n(x) - f_{n+1}(x)| < 2^{-n}$ for every $x \in [0,1]$;

• for every $n$ there exists a family of intervals $I_{n,1}, \dots, I_{n,s_n}$ with total length less than $2^{-n}$ such that $f_n$ is a constant on every interval from the set $[0,1] - (I_{n,1} \cup \ldots \cup I_{n,s_n})$.

(c) Construct a continuous surjective function $f : [0,1] \to [0,1]$ such that for every $\varepsilon > 0$ there exists a family of intervals $I_1, \dots, I_n$ with total length less than $\varepsilon$ such that $f$ is a constant on every interval from the set $[0,1] - (I_1 \cup \ldots \cup I_n)$.

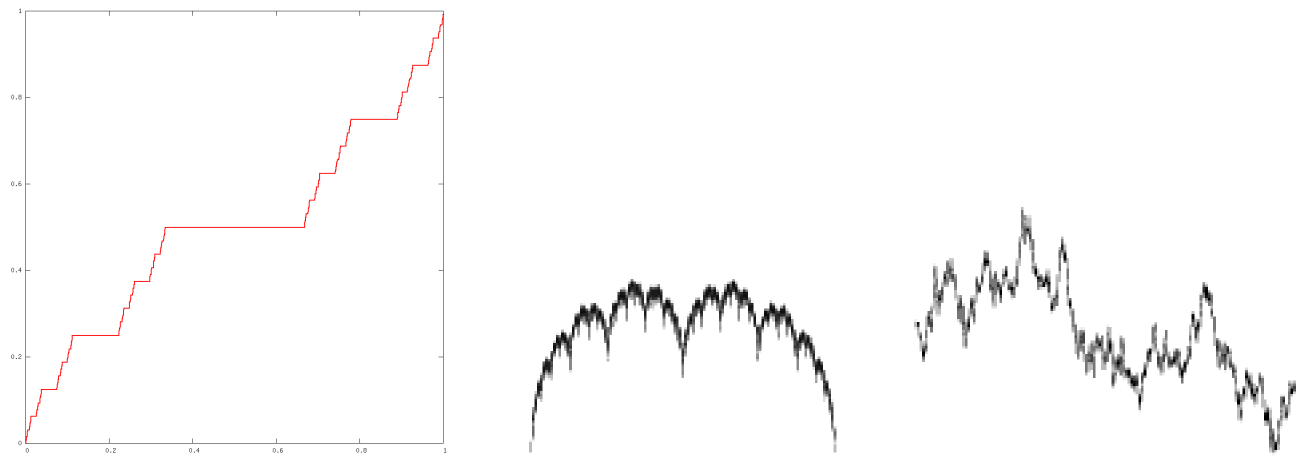


Рис. 1: Cantor function, Weiestrass function and Brownian motion

**4.4.** A sequence of real numbers $\{x_n\}$ is called *fundamental* if for every $\varepsilon > 0$ there exists a natural number $N$ such that for every $m, n > N$ we have $|x_m - x_n| < \varepsilon$.

(a) Is the sequence $x_n = \frac{1}{n}$ fundamental? Is the sequence $x_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$ fundamental?

(b) Every covergent sequence is fundamental.

(c) Every fundamental sequence is bounded.

(d) Every bounded sequence contains covergent subsequence.

(e) Every fundamental sequence has a finite limit.

*If terms which used in the some problem are not defined here and are unknown for you, then you should ignore this problem.*

**4.5.** * There exists a function $f : [0,1] \to [0,1]$ such that continuous everywhere but differentiable nowhere. (Such examples are occured in physics when studying the *Brownian motion*).

**4.6.** (a) Suppose that $f_n : [0,1]^2 \to \mathbb{R}$ is a sequence of functions and every $f_n$ does not depend on the variable $y$. Let $f : [0,1]^2 \to \mathbb{R}$ be the uniform limit of $f_n$. Then the function $f$ does not depend on the variable $y$.

(b)* Let $K := [-1,1]^2 - [-1,0] \times 0$. Is there a function $f : K \to \mathbb{R}$ which *depends* on $y$ such that its restriction on every square which lies in $K$ *does not depend* on $y$?

## 5 Peano curve

**5.1.** A sentence is written in cells of the table following by the unknown rule. Find this rule and read the sentence.

**5.2.** Cells of the square $n \times n$ are indexed by numbers from 1 to $n^2$. Every cell is divided by 4 parts. Prove that an indexing of new splitting $2n \times 2n$ may be chosen such that for the large cell with index $k$ four its subcells have indexes $4k-3, 4k-2, 4k-1, 4k$.

**5.3.** A mapping $f : [a,b] \to \mathbb{R}^2$ is called *linear* if we have

$$f(\lambda x + (1-\lambda)y) = \lambda x + (1-\lambda)y \quad \text{for every} \quad \lambda \in [0,1],\ x, y \in [a,b].$$

A mapping $f : [a,b] \to \mathbb{R}^2$ is called *piecewise linear* if there are points $x_0 = 0, x_1, \ldots, x_n = 1 \in [0,1]$ such that $f$ is linear for every interval $[x_i, x_{i+1}]$.

(a) There exists a piecewise linear mapping $F : [0,1] \to [0,1]^2$ such that for every point $y \in [0,1]^2$ there exists a point $x \in [0,1]$ such that $|y - F(x)| < \frac{1}{100}$.

(b) A piecewise linear mapping $F : [0,1] \to [0,1]^2$ is called $d-$*dense* if for every point $y \in [0,1]^2$ there exist a point $x \in [0,1]$ such that $|y - F(x)| < d$. Prove that for every $d-$dense mapping $F$ there exists $d/2-$dense mapping $F^+$ with condition $|F(x) - F^+(x)| < \frac{1}{100}$ for every $x \in [0,1]$.

(c) There exists a continuous mapping $F : [0,1] \to [0,1]^2$ such that for every point $y \in [0,1]^2$ there exists $x \in [0,1]$ such that $F(x) = y$.

**5.4.** (a) The intersection of any sequence of nested rectangles on the plain is not empty.

(b) — (e) Solve problems 4.4 (b) — (e) for sequences of points on the plain.

## Hints, solutions, and answers

**1.1.** (a) Follows from (b).

(b) Let the vertices of the rectangle have coordinates $(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)$. Then the sum of numbers in opposite vertices is $\varphi_{i_1} + \varphi_{i_2} + \psi_{j_1} + \psi_{j_2}$.

(c) Similar to (b).

(d) The statement is true.

**1.2.** (a) Prove that numbers $i + \sqrt{2}j$ are different for different cells.

(b) (c) Similar to (a).

(d) Yes. Put $\varphi_{ik} = 9i + k$, $\psi_{jk} = 9j + k$.

**1.3.** (a) Prove that each polynomial has only a finite number of roots.

(b) Use the identity $\sin 2x = 2 \sin x \cos x$.

(c) $f(x) = x \sin 1$.

(d) Construct a polynomial that is nonzero only on one value of $x$.

**1.4.** (b) Construct a polynomial that is nonzero only on one set of variables.

**1.6.**(a) Use 1.7.a.

(c) $\overline{\{x + y\}} = \{ax + by \mid a, b \in \mathbb{Z}, a, b \geqslant 0\}$.

(d) $2xy = (x + y)^2 - x^2 - y^2$.

(e) $xy = 2^{\log_2 x + \log_2 y}$.

(f) Use the problem 1.7 (b).

(g) Use the problem 1.3 (a).

**1.7.** (b) Prove that the set of elementary superpositions on $F$ is at most countable.

**1.8.** For each $x \in [0, 1]$ by $0.x_1x_2\dots$ denote the decimal expansion of $x$ such that there is no $N$ such that $x_n = 0$ for each $n > N$.

(a) Use the decimal expansion of $x$.

By definition, put

$$\alpha : [0, 1]^2 \to [0, 1] \quad \text{by} \quad \alpha(x, y) := 0, x_1y_1x_2y_2x_3\dots.$$

Then $\alpha$ is injective (but not bijective!).

(b,c,d) For given $f(x_1, \dots, x_{n+1})$ find $g \in F_n$ such that

$$f(x_1, \dots, x_{n+1}) = g(x_1, \dots, x_{n-1}, \alpha(x_n, x_{n+1})).$$

Notice that $\alpha \in F_n$ for any $n > 1$.

(e) Similarly to (a) construct an injection $[0, 1]^n \to [0, 1]$.

Then (e) follows from (d) and (b) by induction.

**1.9.** (a) $xy = (x + y)^2/4 - (x - y)^2/4$.

(b) $(x + 1)(y + 1) = 2^{\log_2(x+1)+\log_2(y+1)}$

**1.10.** (a) follows from (b), (c) follows from (a) and 1.8.d.

(b) Use the construction from 1.8.b. Take

$$\varphi(x) := 0.x_1 0 x_2 0 x_3 \dots.$$

Then $\alpha(x, y) = \varphi(x) + 0.1\varphi(y)$. Here $\alpha$ is the function which is defined in 1.8.a.

(d) By definition, put

$$\varphi(x) := 0.x_1 \underbrace{0000\dots0}_{n-1 \text{ zeros}} x_2 \underbrace{0000\dots0}_{n-1 \text{ zeros}} x_3 00\dots$$

and $\alpha(x_1, \dots, x_n) := \varphi(x_1) + 0.1\varphi(x_2) + \dots + (0.1)^{n-1}\varphi(x_n)$. Then $\alpha$ is injective. Hence, $h \in F_1$ exists for every $f \in F_n$.

**2.1.** (b) Prove that $n^2 - 3n + 5 > n$ for all $n > 4$.

(c) Use 2.2.

**2.2.** Use the induction by $k$.

**2.3.**. (a) take $a = -\frac{1}{2}$

(b) $$\sqrt{5 + \frac{2}{n}} - \sqrt{5} = \frac{\left(\sqrt{5 + \frac{2}{n}} - \sqrt{5}\right)\left(\sqrt{5 + \frac{2}{n}} + \sqrt{5}\right)}{\sqrt{5 + \frac{2}{n}} + \sqrt{5}} = \frac{2}{n\left(\sqrt{5 + \frac{2}{n}} + \sqrt{5}\right)}$$

(c) take $a = 0$ and use Bernoulli's inequality.

(d) take $a = 1$ and use Bernoulli's inequality.

(e) take $a = 0$ and find $k$ such that $(n+1)^9/n^9 < 1.5$ for $n > k$.

**2.4.** (a) Clearly, if $|x| < 1$ then $(x-3)^2 < 16$ and $|3(x-3)| < 12$. Then

$$|(x-3)^3 + 3^3| = |x((x-3)^2 - 3(x-3) + 3^2)| \leqslant |x| \cdot |16 + 12 + 9| < 40|x|$$

(c) Use the inequality $\sin x < x$.

(d) If

$$|f(x) - a| < \varepsilon/2 \quad \text{for} \quad x \in (-\delta_1, \delta_1) \quad \text{and}$$
$$|g(x) - b| < \varepsilon/2 \quad \text{for} \quad x \in (-\delta_2, \delta_2),$$
$$\text{then} \quad |f(x) + g(x) - a - b| < \varepsilon \quad \text{for} \quad x \in (-\min\{\delta_1, \delta_2\}, \min\{\delta_1, \delta_2\}).$$

The same inequality is true for $f - g$. Statements for $fg$ and $f/g$ can be proven in a similar way.

**3.2.** (a) The function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ is continuous. Take $\delta = \varepsilon$ and use the triangle inequality $|f(z) - f(z_0)| \leqslant |z - z_0|$.

(b) Let us prove that $f$ is not continuous. Take $x = 1$, $y = 0$, and $\varepsilon = \frac{1}{2}$. Then $|f(1,0) - f(1 - \frac{\delta}{2}, 0)| = 1 > \frac{1}{2}$ for every $\delta > 0$.

**3.3.** (a) By definition of continuity there exists $N$ such that if $|x - y| < 1/N$ then $|f(x) - f(y)| < 1$. Then

$$|f(x)| < 1 + \max\left\{|f(0)|, \left|f\left(\frac{1}{N}\right)\right|, \dots, \left|f\left(\frac{N-1}{N}\right)\right|, |f(1)|\right\}$$

for every $x \in [0,1]$.

(b) By (a) there exists a supremum of $f$ on $[0,1]$, say $u$. We call a subset $A \subseteq [0,1]$ *good* if $u$ is supremum of $f$ on $A$. Clearly, if interval $[a,b]$ is good, then at least one of the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ is good. Now starting from $I_0 = [0,1]$ we construct a sequence of good intervals $\{I_n\}$ such that $I_{n+1} \subset I_n$ and $|I_n| = 2^{-n}$ for each $n > 0$. Then $f$ reaches its maximal value at the intersection point $\bigcap_n I_n$.

**3.4.** (a) Yes. Choose $\delta = \varepsilon/2$.

(b) Yes. There exists $C$ such that $|f(x)|, |g(x)| < C$ for each $x \in [0,1]$. This follows from 3.3.a. Take $\delta$ such that $\delta^2 + 2\delta C < \varepsilon$.

(c) Yes. Let us show that $f \circ g$ is continuous. Take $\delta_1$ such that if $|x - y| < \delta_1$ then $|f(x) - f(y)| < \varepsilon$. Further, take $\delta$ such that if $|x - y| < \delta$ then $|g(x) - g(y)| < \delta_1$.

(d) Take $\delta_1 > 0$ such that if $|x_1 - x_2| < \delta_1$ and $|y_1 - y_2| < \delta_1$ then $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$. Take $\delta_2$ such that if $|x_1 - x_2| < \delta_2$ then $|g(x_1) - g(x_2)| < \delta_1$. Further, take $\delta = \min(\delta_1, \delta_2)$.

(e) Similar to (d).

**3.5.** There exists such function. Define $f(x,y) = x/y$ for $x < y$, $f(x,y) = y/x$ for $y \leqslant x$, and $f(0,0) = 0$. This function is not continuous because for each $\varepsilon > 0$ we have $f(\varepsilon, \varepsilon) - f(0,0) = 1$.

**4.1.** (a) To show that $\lim_{n\to\infty}\sum_{k=0}^{n} x^k = \frac{1}{1-x}$, estimate

$$\left|\frac{1}{1-x} - \sum_{k=0}^{n} x^k\right| = \frac{x^n}{1-x} = \frac{1}{(1-x)(1+(1/x-1))^n} < \frac{1}{(1-x)n(1/x-1)}$$

by Bernoulli's inequality.

(b) No, because for each $n$ there exists $x \in (0,1)$ such that $\frac{x^n}{1-x} > 0.5$.

**4.2.** (a) Suppose $f_n(x) = x^n$. Then $f(x) = 0$ for $x < 1$ and $f(1) = 1$.

(b) Yes. Suppose $|f_n(x) - f(x)| < \varepsilon$ if $n > N$. Choose $\delta$ such that $|f_{N+1}(x) - f_{N+1}(y)| < \varepsilon$ for $|x - y| < \delta$. Then $|f(x) - f(y)| < 3\varepsilon$ for $|x - y| < \delta$.

**4.3.** (a) Take $f$ linear on intervals $[0, 0.99]$ and $[0.99, 1]$; $f(0) = f(0.99) = 0$, $f(1) = 1$.

(b) Construct a sequence of families of intervals $I_k^i$ and piecewise linear functions $\{f_k\}$ with following conditions:

1) $f_k$ is linear on each interval $I_k^i$;
2) number of intervals in the family $I_k^i$ is $2^k + 1$;
3) $f_k$ is constant on each odd interval;
4) $f_k$ is increasing on each even interval.

Let $f_1$ be the function from (a). Then other $f_k$ are defined by induction in a following way. If $f_k$ is constant on $I_k^i$, then $f_{k+1} \equiv f_k$ on $I_k^i$. If $f_k$ increases on $I_k^i = [t_1, t_2]$, $f(t_1) = a$, $f(t_2) = b$, then

$$f_{k+1}(t_1) = a,\, f_{k+1}(t_2) = b,\, f_{k+1} \equiv (a+b)/2 \text{ on } [\frac{2t_1+t_2}{3}, \frac{t_1+2t_2}{3}],$$

and $f_{k+1}$ is linear on intervals $[t_1, \frac{2t_1+t_2}{3}]$ and $[\frac{2t_1+t_2}{3}, t_2]$.

(c) For $x \in [0,1]$ the sequence $\{f_k(x)\}$ is bounded. From the principle of nested intervals it follows that this sequence has a *limit point*, say $f(x)$. Let us recall that a limit point of sequence $\{y_n\}$ is a real number $y$ such that for each $N$ and $\varepsilon$ there exists $n > N$ such that $y_n \in (y-\varepsilon, y+\varepsilon)$. The function $f(x)$ is the uniform limit of $\{f_k\}$. One can easily ti see that all properties discussed above hold.

Remark. Another proof is based on ternary expansion.

**4.5.** The classical example is *Weierstrass function*

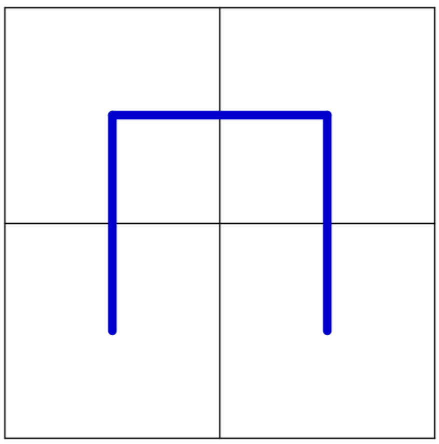
$$f(x) = \sum_{n=0}^{\infty} 2^{-n}\sin(13^n \pi x),$$

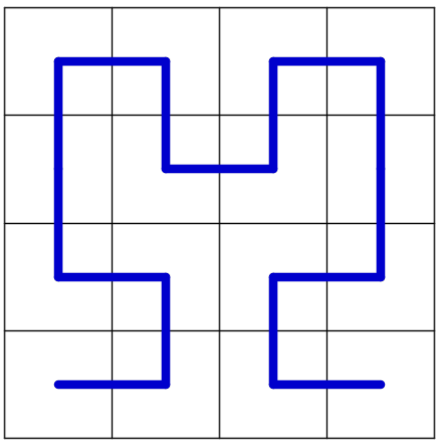
see fig. 1.

Remark. Another solution is based on Baire's theorem.

**5.1.** The text can be read along the curve invented by italian mathematician Peano.

**5.2.**





**5.3.** (a) Construct required map as «indexing» of cells of decomposition of square $[0, 1]^2$ into $n^2$ cells.

(b) The image $F[0, 1]$ of $F$ is a polygonal line. Construct $F^+$ such that the following conditions hold:

this image is a result of substitution of links $F[0, 1]$ to a polygonal lines;

polygonal line which substitute link is $d/2-$dense in the $d$-neighbourhood of this link.

(c) Construct required map as a uniform limit of functions which are constructed in (b).

**5.4.** (a) Consider the sequence $a_i = (x_i, y_i)$ of centers of recatngles. Prove that any bounded sequence of points have a *limit point*. Any limit point of the sequence $a_i$ lies in each rectangle.

## 6 Deduction of the Kolmogorov theorem from approximative version

The aim is to show that any *continuous* multivariable function can be represented as a superposition of several *continuous* functions of one variable and *the addition.* We consider functions of two variables on the unit square.

In fact, Kolmogorov's theorem 1.11 is a statement about approximation of given continuous function by continuous functions of a special type. Main ingredient of proof is approximative Kolmogorov's theorem 6.2.b. This theorem allows us to represent any continuous function as a sum of series. All summands in this series have a required type. Moreover, these summands can be arranged in a special way.

Let us recall that $I := [0,1]$ and $I^2 := [0,1] \times [0,1]$.

Now let us prove Kolmogorov's theorem. We use the approach [Ka] (see also [He]) and combine Kolmogorov's ideas and Baire theorem.

Let $f : I^2 \to \mathbb{R}$ be an arbitrary function. **Kolmogorov set** for $f$ is a set of continuous functions $\varphi_1, \varphi_2, \dots, \varphi_5 : I \to I$, $h : [0,3] \to \mathbb{R}$ such that

$$f(x,y) = h\left(\varphi_1(x) + \sqrt{2}\varphi_1(y)\right) + \ldots + h\left(\varphi_5(x) + \sqrt{2}\varphi_5(y)\right).$$

for any $x, y \in I$.

**6.1.** Construct a Kolmogorov set for the function $f(x,y) = x + \sqrt{2}y + 3$.

We do not know explicit Kolmogorov set even for the addition and the multiplication.

For any $f : I^2 \to \mathbb{R}$ let us denote

$$|f| := \max_{z \in I^2} |f(z)|.$$

Let $f : I^2 \to \mathbb{R}$ be an arbitrary function and $\lambda$ be a number. **Kolmogorov set** for $(f, \lambda)$ is a set of continuous functions $\varphi_1, \dots, \varphi_5 : I \to I$, $h : [0,3] \to \mathbb{R}$ such that $|h| \leqslant 2(1-\lambda)|f|$ and

$$\left| f(x,y) - \sum_{k=1}^{5} h\left(\varphi_k(x) + \sqrt{2}\varphi_k(y)\right) \right| \leqslant \lambda |f|$$

for any $x, y \in I$.

**6.2.** (a) Construct a Kolmogorov set for $f(x,y) = xy$ and $\lambda = 1/2$.

(b)* Let $f$ be an arbitrary continuous function. Then there exists a Kolmogorov set for $f$ and $\lambda = 5/6$.

(c)* **Approximative Kolmogorov's theorem.** *There exists a set of continuous functions* $\varphi_1, \dots, \varphi_5 : I \to \mathbb{R}$ *with the following property: for an arbitrary continuous function* $f : I^2 \to \mathbb{R}$ *and arbitrary small* $\lambda > 0$ *there exists a continuous function* $h : [0,3] \to \mathbb{R}$ *such that* $\{\varphi_1, \dots, \varphi_5, h\}$ *is a Kolmogorov set for* $(f, \lambda)$.

We stress again that (b), (c) are not easy (see §7-§9).

First of all, let us explain how these results can be applied.

Approximative Kolmogorov's theorem allows us to *approximate* any continuous function $f$ by linear combination of functions of type $h(\varphi(x) + \sqrt{2}\varphi(y))$. Moreover, «inner» function $\varphi$ does not depend on $f$. Hence, we get required representation (i.e. Kolmogorov set for $f$) and arrange summands in some series naturally constructed by Kolmogorov sets for $(f, \lambda)$.

**6.3.** Let a set $\{\varphi_1, \dots, \varphi_5, h\}$ be a Kolmogorov set for $f, \lambda$ and

$$f_1(x,y) := f(x,y) - \sum_{k=1}^{5} h\left(\varphi_k(x) + \sqrt{2}\varphi_k(y)\right).$$

(a) Suppose a set $\{\varphi_1, \dots, \varphi_5, h_1\}$ is a Kolmogorov set for $(f_1, \lambda)$. Then the set $\{\varphi_1, \dots, \varphi_5, h+h_1\}$ is a Kolmogorov set for $(f, \lambda^2)$.

(b) Get approximative Kolmogorov's theorem from similar statement «for $\lambda = 5/6$».

(c) Get the Kolmogorov's theorem from the approximative Kolmogorov's theorem.

## 7 Lemma of approximation by prekolmogorov sets

Suppose $f$ is a function $I^2 \to \mathbb{R}$. An ordered set of continuous functions $\varphi_1, \dots, \varphi_5 : I \to \mathbb{R}$ is called a **prekolmogorov set** for $f$ if there exists a continuous function $h : [0, 3] \to \mathbb{R}$ such that the set $\varphi_1, \dots, \varphi_5, h$ is a Kolmogorov set for $f$ with $\lambda = 5/6$.

Approximative Kolmogorov's theorem «for $\lambda = 5/6$» claims that there exists an ordered set $\varphi_1, \dots, \varphi_5 : I \to \mathbb{R}$ such that it is prekolmogorov set for any continuous function $f : I^2 \to \mathbb{R}$. Another way to construct such a set is an iteration approximation process or, in more formal words, the Baire category theorem.

Let $M \subset \mathbb{R}^2$ be an arbitrary set. Two functions $\varphi, \psi : M \to \mathbb{R}$ are called $\varepsilon$**-close**, if $|\psi(x) - \varphi(x)| < \varepsilon$ for any $x \in M$. Two ordered sets of functions $(\varphi_1, \dots, \varphi_5 : M \to \mathbb{R})$ and $(\psi_1, \dots, \psi_5 : M \to \mathbb{R})$ are called $\varepsilon$**-close**, if $\varphi_k, \psi_k$ are $\varepsilon$-close for each $k = 1, \dots, 5$.

**7.1.** * (a) **Prekolmogorov stability lemma.** Suppose a set $(\varphi_1, \dots, \varphi_5)$ is a prekolmogorov set for $f : I^2 \to \mathbb{R}$. Then there exists $\varepsilon > 0$ such that any $\varepsilon$-close to $(\varphi_1, \dots, \varphi_5)$ set is a prekolmogorov set for$f$.

(In other words, the set $PK(f)$ of all prekolmogorov sets for $f$ is open in the space $C(I^2)^5$. Here $C(I^2)$ is the space of all continuous functions $I^2 \to \mathbb{R}$.)

(b) **Lemma of approximation by prekolmogorov sets.** For any $\varepsilon > 0$ and continuous functions $f : I^2 \to \mathbb{R}$, $\psi_1, \dots, \psi_5 : I \to \mathbb{R}$ there exists a prekolmogorov set for $f$ such that it is $\varepsilon$-close to $(\psi_1, \dots, \psi_5)$.

(This means that $PK(f)$ is dense in $C(I^2)^5$.) Hints are provided in §8.

## 8 Proof of the approximation lemma

**8.1.** (a) For any numbers $\psi_1, \dots, \psi_8 \in I$ there are $\varphi_1, \dots, \varphi_8 \in I$ such that

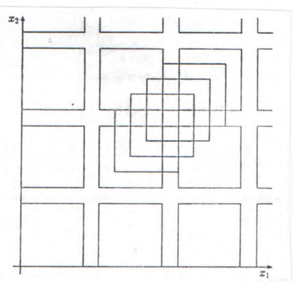
- $|\psi_i - \varphi_i| < 0.01$ for every $i = 1, \dots, 8$.
- for any arrangement of numbers on the chessboard there exists a continuous function $h : [0, 3] \to \mathbb{R}$ such that a number at square $(i, j)$ is equal to $h(\varphi_i + \sqrt{2}\varphi_j)$.

(b) For any numbers $\psi_{i,k} \in I$, $i, k = 1, \dots, 8$ there are numbers $\varphi_{i,k} \in I$, $i, k = 1, \dots, 8$ such that

- $|\psi_{i,k} - \varphi_{i,k}| < 0.01$ for every $i, k = 1, \dots, 8$.
- for any arrangement of numbers in cells of the cube $8 \times 8 \times 8$ there exists a continuous function $h : [0, 3] \to \mathbb{R}$ such that the number in the cell $(i, j, k)$ is equal to $h(\varphi_{i,k} + \sqrt{2}\varphi_{j,k})$.

**8.2.** (a) Let $x \in \mathbb{R}$ be an arbitrary point. Then for every $k = 1, \dots, 5$ *except at most one* there exists $j \in \mathbb{Z}$ such that $x$ lies in the interval $4I + 5j + k := [5j + k, 4 + 5j + k]$.

(b) Let $(x, y) \in \mathbb{R}^2$ be an arbitrary point. Then for every $k = 1, \dots, 5$ *except at most two* there are $i, j \in \mathbb{Z}$ such that $(x, y)$ lies in the square $(4I + 5i + k) \times (4I + 5j + k)$.



Visual interpretation of this statement is following. For each $k = 1, \dots, 5$ draw these squares on the plain. We have a «city map» of $k$-th city with *blocks* (squares) and *streets* (intervals between squares). Let us impose city maps for five cities one on another. Then every point lies inside of block at least for three of city maps.

(a',b') Formulate and prove analogues of (a,b) with change 5 to an arbitrary positive integer.

We say that a function $\varphi : I \to \mathbb{R}$ *rationally separates* the family of pairwise not crossed intervals on the line if the following conditions hold:

- $\varphi$ is continuous;
- $\varphi$ is a constant on every interval from this family;
- $\varphi$ has only rational values which are different for different intervals.

**8.3.** Let $\psi : I \to \mathbb{R}$ be a continuous function and $\varepsilon > 0$. Then there exists an integer $N > 0$ and function $\varphi : I \to \mathbb{R}$ such that $\varphi$ is $\varepsilon$-close to $\psi$ and the following condition holds:

(a) $\varphi$ is a constant on every interval $\left[\frac{j-1}{N}, \frac{j}{N}\right)$ for each $j \in \{1, 2, \dots, N\}$.

(b) $\varphi$ is continuous and linear on every interval $\left[\frac{j-1}{N}, \frac{j}{N}\right]$ for each $j \in \{1, 2, \dots, N\}$.

(c) $\varphi$ rationally separates the family of intervals $\frac{4I+5j}{N} := \left[\frac{5j}{N}, \frac{4+5j}{N}\right]$ for each $\frac{4I+5j}{N}$, $j \in \left\{0, 1, \dots, \frac{N-4}{5}\right\}$.

We say that a function $\varphi : I^2 \to \mathbb{R}$ *separates* the family of pairwise not crossed squares on the plain if the folloing conditions hold:

- $\varphi$ is continuous;
- $\varphi$ is a constant on every square from this family;
- $\varphi$ has different values for different squares.

**8.4.** Suppose $\varphi : [0, 1000] \to \mathbb{R}$ rationally separates the family of intervals $4I + 5j := [5j, 4 + 5j]$, $j \in \{1, \dots, 100\}$ (the definition is similar to the case when $\varphi : I \to \mathbb{R}$.) Then

(a) Function $\varphi(x) + \sqrt{2}\varphi(y)$ separates the family of squares $(4I + 5i) \times (4I + 5j)$, $i, j \in \{1, \dots, 100\}$. (the definition is similar to the case when $\varphi : I^2 \to \mathbb{R}$).

(b) For any set $v_{i,j}$ of numbers, $i, j \in \{1, \dots, 100\}$ there exists a continuous function $h : [0, 3] \to \mathbb{R}$ such that

- $|h(x)| \leqslant \max_{i,j} |v_{ij}|$ for every $x \in [0, 3]$.
- $h(\varphi(x) + \sqrt{2}\varphi(y)) = v_{i,j}$ for every $i, j \in \{1, \dots, 100\}$, $x \in 4I + 5i$, $y \in 4I + 5j$.

**8.5.** Suppose for every $k \in \{1, \dots, 5\}$ function $\varphi_k : [0, 1000] \to I$ rationally separates the family of intervals $4I + 5j + k := [5j + k, 4 + 5j + k]$, $j \in \{1, \dots, 100\}$ and numbers $\varphi_k(5j + k)$ are different for different pairs $(j, k)$. Then for any set $v_{k,i,j}$ of numbers $k \in \{1, \dots, 5\}$, $i, j \in \{1, \dots, 100\}$ there exists a continuous function $h : [0, 3] \to \mathbb{R}$ such that $h(\varphi_k(x) + \sqrt{2}\varphi_k(y)) = v_{k,i,j}$ for every

$$k \in \{1, \dots, 5\}, \quad i, j \in \{1, \dots, 100\}, \quad x \in 4I + 5i, \quad y \in 4I + 5j.$$

**8.6.** Let $f : I^2 \to \mathbb{R}$ be a function. Choose an integer $N > 0$, $N \equiv 4 \mod 5$ such that

$$|f(z) - f(z')| < \frac{|f|}{6} \quad \text{for every} \quad i, j \in \left\{0, 1, \dots, \frac{N-4}{5}\right\}, \quad z, z' \in \frac{4I+5i}{N} \times \frac{4I+5j}{N}.$$

If functions $\varphi_1, \dots, \varphi_5 : I \to I$ separate the family of intervals

$$\frac{4I+5j}{N}, j \in \left\{0, 1, \dots, \frac{N-4}{5}\right\}$$

then

(a) If $h : [0, 3] \to \left[-\frac{|f|}{3}, \frac{|f|}{3}\right]$ is a continuous function and

$$h\left(\varphi_k(x) + \sqrt{2}\varphi_k(y)\right) = \frac{1}{3} f\left(\frac{5i+k}{N}, \frac{5j+k}{N}\right) \quad \text{for every}$$

$$k = 1, \dots, 5, \quad i, j \in \left\{0, 1, \dots, \frac{N-4}{5}\right\}, \quad x \in \frac{4I + 5i}{N}, \quad y \in \frac{4I + 5j}{N},$$

then the set $\varphi_1, \dots, \varphi_5, h$ is 5/6-kolmogorov set for $f$.

(b) The set $\varphi_1, \dots, \varphi_5$ is a prekolmogorov set for $f$.

(c) (Riddle) Formulate the analogue of (a) with change 5 to 4. Is it true?

**Hints, solutions, and answers for problems after the Semifinal**

For $\varphi : [0,1] \to \mathbb{R}$ denote

$$\widetilde{\varphi}(x,y) := \varphi(x) + \sqrt{2}\varphi(y) \quad \text{и} \quad z = (x,y).$$

**6.1.** Put $\varphi_1(x) = x$, $\varphi_2(x) = \varphi_3(x) = \varphi_4(x) = \varphi_5(x) = 0$, $h(x) = x + \frac{3}{5}$

**6.2.** (b,c) Follows from 9.1.b and 9.2.ac.

**6.3.** (b) We have $|h + h_1| < 2|f|(1 - \lambda + \lambda(1-\lambda)) = 2|f|(1-\lambda^2)$ and

$$\left|f(z) - \sum_{k=1}^{5}(h+h_1)(\widetilde{\varphi}_k(z))\right| = \left|f_1(z) - \sum_{k=1}^{5} h_1(\widetilde{\varphi}_k(z))\right| < \lambda|f_1| < \lambda^2|f|.$$

(c) Repeat the process from (b). We get appoximative Kolmogorov's theorem for $\lambda = (5/6)^{2^n}$ by induction.

(d) Repeat the process from (b). Construct a sequence of functionf $h_n : [0,3] \to \mathbb{R}$ such that

$$\left|f - \sum_{k=1}^{5}(h_0 + h_1 + \ldots + h_n)(\widetilde{\varphi}_k(z))\right| \leqslant \left(\frac{5}{6}\right)^{n+1}|f| \quad \text{and} \quad |h_n| < \frac{1}{3}\left(\frac{5}{6}\right)^{n}|f|$$

for each $n$. $\sum\limits_n h_n$ is uniformly converge to $h$. This follows from second inequality. Take $n \to \infty$ in first inequality. We have $f(z) = \sum\limits_{k=1}^{5} h(\widetilde{\varphi}_k(z))$.

**7.1.** (b) Use 8.3.c. Find $N \equiv 4 \mod 5$ and functions $\varphi_1, \ldots, \varphi_5$ such that for each $k$ function $\varphi_k$ is close to $\psi_k$ and separates families of intervals

$$\frac{4I + 5j + k}{N}, j = 1, \ldots, \frac{N-4}{5}.$$

Similarly to 8.5 find continuous function $h : [0,3] \to \mathbb{R}$ such that

$$h(\widetilde{\varphi}_k(x,y)) = \frac{1}{3}f\left(\frac{5i+k}{N}, \frac{5j+k}{N}\right)$$

for any

$$k = 1, \ldots, 5, i, j = 1, \ldots, \frac{N-4}{5}, x \in \frac{4I + 5i + k}{N}, y \in \frac{4I + 5j + k}{N}.$$

After that let us increase $N$ (i.e. decrease squares). We can take $N$ sufficiently large such that condotions of 8.6 are satisfied. Then the Lemma follows from 8.6.b.

**8.1.** (a) Take pairwise different rational numbers $\varphi_1, \ldots, \varphi_8$ that are 0.01-close to $\psi_1, \ldots, \psi_8$ (i.e. $|\psi_i - \varphi_i| < 0.01$ for each $i$). If $p + \sqrt{2}q = s + \sqrt{2}t$ for rational $p, q, s, t$, then $p = s$ and $q = t$. Then $\varphi_i + \sqrt{2}\varphi_j$ are different numbers for different pairs $(i,j)$. Define $h(\varphi_i + \sqrt{2}\varphi_j)$ to be the number in cell $(i,j)$, and extend $h$ piecewise linearly to $\mathbb{R}$.

(b) Take pairwise different rational numbers $\varphi_{i,k}$ such that $|\psi_{i,k} - \varphi_{i,k}| < \varepsilon$. Then the equality $\varphi_{i,k} + \sqrt{2}\varphi_{j,k} = \varphi_{s,m} + \sqrt{2}\varphi_{t,m}$ implies that $(i,k) = (s,m)$ and $(j,k) = (t,m)$, that is $(i,j,k) = (s,t,m)$.

**8.2.** (a) Let $m = [x/5]$ and $r = [x] \bmod 5$. Then $x \in [5m + r, 5m + r + 1)$. Clearly, $x \in [5m + (r - s), 5m + (r - s + 4)]$ for each $s \in \{0,1,2,3\}$.

(b) This follows from (a).

**8.3.** (c) Take a number $N$ such that for each $x, y \in I$ the inequality $|x-y| < 5/N$ implies $|\psi(x)-\psi(y)| < \varepsilon/4$. Take pairwise distinct rational numbers $q_j$, $j \in \mathbb{Z}$, such that $|q_j-\psi(x)| < \varepsilon/3$ for each $j \in \left\{0, 1, \ldots, \frac{N-4}{5}\right\}$ and $x \in \frac{4I+5j}{N}$. Define $\varphi(x) = q_j$ if $x \in \frac{4I+5j}{N}$, and extend $\varphi$ piecewise linearly to $I$. Then $|\psi(x) - \varphi(x)| < \varepsilon$ for each $x \in I$.

**8.4.** (b) Denote $\alpha_{i,j} := \varphi_i(x) + \sqrt{2}\varphi_j(y)$ for some $x \in 4I+5i$ and $y \in 4I+5j$. If $\alpha_{i,j} = \alpha_{k,l}$, then $(i,j) = (k,l)$. Define $h(\alpha_{i,j}) = v_{i,j}$, and extend $h$ piecewise linearly to $\mathbb{R}$.

**8.5.** Similarly to 8.4.b.

**8.6.** (a) Take any point $z \in I^2$. Similarly to 8.2.b for any $k = 1, \ldots 5$, except at most two, there exist

$$i, j \in \left\{0, 1, \ldots, \frac{N-4}{5}\right\}$$

such that

$$z \in \frac{4I+5i+k}{N} \times \frac{4I+5j+k}{N}.$$

Without loss of generality it is true for $k = 1, 2, 3$. Then

$$\left|f(z) - \sum_{k=1}^{5} h(\widetilde{\varphi}_k(z))\right| \leqslant \sum_{k=1}^{3} \left|\frac{1}{3}f(z) - h(\widetilde{\varphi}_k(z))\right| + \sum_{k=4}^{5} |h(\widetilde{\varphi}_k(z))| < 3 \cdot \frac{1}{3} \cdot \frac{|f|}{6} + 2 \cdot \frac{|f|}{3} = \frac{5|f|}{6}.$$

(b) Follows from (a).

## 9 Deduction of approximative Kolmogorov's theorem from lemma of approximation

For $M \in \{I, I^2\}$ let us denote by $C(M)$ the set of continuous functions $M \to \mathbb{R}$.

**9.1.** (a) There exists countable dense subset in $C(I)$.

(b) There exists countable dense subset in $C(I^2)$.

**9.2.** Let $f_l : I^2 \to \mathbb{R}$ be a dense subset in $C(I^2)$. Then

(a) Any set from $\bigcap\limits_{l=1}^{\infty} PK(f_l)$ is prekolmogorov set for any continuous function $f : I^2 \to \mathbb{R}$.

(b) For $C(I^2)^5$ the «principle of nested balls» is satisfied and Baire's theorem holds. (This means that $C(I^2)^5$ is *complete metric space*.)

(c) $\bigcap\limits_{l=1}^{\infty} PK(f_l) \neq \emptyset$.

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