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# 1 Opening problems

For a start, you can try to solve several problems. They are rather complicated, so if you wouldn't succeed, return to this section after studying $pqr$-lemmas.

**1.** Let $a$, $b$, $c$ be non-negative real numbers such that $a+b+c=1$. Prove that

$$1+12abc \geqslant 4(ab+bc+ca).$$

**2.** Let $a$, $b$, $c$ be real numbers such that

$$a+b+c=9, \quad ab+bc+ca=24.$$

Prove that $16 \leqslant abc \leqslant 20$. Prove moreover that for any $r \in [16, 20]$ there exist real numbers $a$, $b$, $c$ such that $a+b+c=9$, $ab+bc+ca=24$, $abc=r$.

**3.** Let $P$ be a symmetric polynomial of degree not greater than 5. Prove that if $P(a,a,c) \geqslant 0$ and $P(0,b,c) \geqslant 0$ for all non-negative real numbers $a, b, c$, then $P(a,b,c) \geqslant 0$ for all non-negative real numbers $a$, $b$, $c$.

**4.** (Russia TST, 2015) Let $a$, $b$, $c$ be non-negative real numbers such that $1+a+b+c = 2abc$. Prove that

$$\frac{ab}{1+a+b}+\frac{bc}{1+b+c}+\frac{ca}{1+c+a} \geqslant \frac{3}{2}.$$

**5.** (Iran TST, 1996) Let $a$, $b$, $c$ be non-negative real numbers such that $ab+bc+ca \neq 0$. Prove that

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) \geqslant \frac{9}{4}.$$

# 2 Symmetric polynomials

We start out discussion of $pqr$-method by considering symmetric polynomials. Main results of this section will be used in the next sections.

[1] At the moment the name $uvw$-method is commonly used.

[2] The authors are grateful to Michael Rozenberg for useful remarks. We would like to draw your attention to the important role he played in popularization of the $uvw$-method.

[3] The authors are grateful to Anna Doledenok and Ilya Bogdanov for help with the translation.

## 2.1 Properties of symmetric polynomials

**6.** Express $a^2+b^2$, $a^3+b^3$, $a^4+b^4$, $(a-b)^2$ in terms of $a+b$ and $ab$.

For three variables $a, b, c$ denote $p = a+b+c$, $q = ab+bc+ca, r = abc$ (elementary symmetric polynomials in 3 variables).

**7.** Express polynomials $a^2+b^2+c^2$, $a^2b+a^2c+b^2a+b^2c+c^2a+c^2b$, $a^3+b^3+c^3$, $(ab)^2+(bc)^2+(ca)^2$, $a^4+b^4+c^4$, $(a+b)(b+c)(c+a)$ in terms of $p, q, r$.

**8.** By definition, put $s_k = a^k+b^k+c^k$ for any non-negative integer number $k$. Express $s_k$ $(k \geqslant 3)$ in terms of $p$, $q$, $r$, $s_{k-1}$, $s_{k-2}$ and $s_{k-3}$.

The polynomial $G(a, b, c)$ in 3 variables is called *symmetric* if its value does not change after swapping any two variables (i.e. $G(a, b, c) = G(b, a, c) = \ldots$).

You may use result of the next problem in the sequel without proof.

**9.** Prove that any symmetric polynomial in $a, b, c$ can be expressed as a polynomial in $p$, $q$, $r$.

**10.** Let $t$ be a real number. Solve a system of equations: $\begin{cases} a+b+c=t, \\ a^2+b^2+c^2=t^2, \\ a^3+b^3+c^3=t^3. \end{cases}$

Often it is useful to rewrite an inequality to be proved in terms of symmetric polynomials. However, some troubles are to be resolved. For instance, a trivial inequality $a^2+b^2+c^2 \geqslant 0$, where $a$, $b$, $c$ are real numbers, turns into $p^2 - 2q \geq 0$. This resulting inequality holds not for all pairs of real $p$ and $q$. Thus, it turns out that the triples of $p$, $q$, and $r$ are not arbitrary. Our next goal is to work out the conditions $p$, $q$, and $r$ must satisfy.

**11.** Let $a$, $b$, $c$ be real numbers. Prove that $q^2 \geqslant 3pr$.

**12.** Let $a$, $b$, $c$ be non-negative real numbers. Prove that $\frac{p}{3} \geqslant \sqrt{\frac{q}{3}} \geqslant \sqrt[3]{r}$.

## 2.2 Symmetric polynomials in 2 variables

A *complex number* is an ordered pair of real numbers $(x, y)$. It is useful to write the pair $(x, y)$ as $x+iy$, where $i$ is the *imaginary unit.* Define the sum and the product of complex numbers as:

$$(a+bi)+(c+di) = (a+c)+(b+d)i,$$

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

It is easy to see that $i^2 = -1$. For the complex number $x$ is called *real part* and $y$ is called *imaginary part.* A complex number is called *pure imaginary* if $x = 0$. The *complex conjugate* of a complex number $x+yi$ is the number $x-yi$ (it is denoted as $\overline{x+yi}$). Notice that every real number $x$ is complex: $x = x+0\cdot i$.

For two complex numbers $a$, $b$ denote $p = a + b$ and $q = ab$.

**13.** Prove that $a$ and $b$ are roots of equation $x^2 - px + q = 0$ and there are no other roots.

**14.** Prove that if $p$ and $q$ are real, then either $a$ and $b$ are both real or $b$ is the complex conjugate of $a$.

**15.** Prove that if $p$ and $q$ are real, then $(a - b)$ is either real or pure imaginary.

**16.** Which conditions (particularly, inequalities) should satisfy $p$ and $q$ for $a$ and $b$ to be real?

**17.** Prove that $a$ and $b$ are real and non-negative if and only if $p$ and $q$ are non-negative real numbers which satisfy conditions from the previous problem.

## 2.3 Symmetric polynomials in 3 variables

For three (perhaps complex) numbers $a$, $b$, $c$ denote $p = a+b+c$, $q = ab+bc+ca$, $r = abc$.

**18.** Prove that $a$, $b$, $c$ are roots of equation $x^3 - px^2 + qx - r = 0$ and there are no other roots.

**19.** Prove that for real numbers $p'$, $q'$, $r'$ there exist complex numbers $a'$, $b'$, $c'$ (unique up to a permutation) such that $p' = a' + b' + c'$, $q' = a'b' + b'c' + c'a'$, $r' = a'b'c'$. Prove moreover that either numbers $a'$, $b'$, $c'$ are real or $a'$ is real and $b'$ is a complex conjugate of $c'$ (up to a permutation).

**20.** Assume that $p$, $q$, and $r$ are real numbers. Prove that if $a$, $b$, $c$ are real, then $(a - b)(b - c)(c - a)$ is real, otherwise it is pure imaginary.

**21.** Prove that

$$(a - b)^2(b - c)^2(c - a)^2 = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2.$$

*Hint.* Denote $l_{xy} = a^xb^y + b^xc^y + c^xa^y$. Then $(l_{12} - l_{21})^2 = (l_{12} + l_{21})^2 - 4l_{12}l_{21}$.

**22. Criterion for reality.** Let $(p, q, r)$ be a triple of real numbers. Prove that the numbers $a$, $b$, $c$ (which are defined as the roots of $x^3 - px^2 + qx - r = 0$ counting multiplicities) are real if and only if $T(p, q, r) \geqslant 0$, where $T(p, q, r)$ is a polynomial in 3 variables defined as $T(p, q, r) = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2$.

**23. Non-negativity lemma.** Prove that $p$, $q$, $r \geqslant 0$ and $T(p, q, r) \geqslant 0$ if and only if $a$, $b$, $c$ are non-negative real numbers.

The formula for $T(p, q, r)$ may look too complicated for being useful. In what follows, we shall mainly use the following idea. Let us regard the expression $T(p, q, r)$ as a polynomial in one variable (e.g., $r$), the other two are assumed to be fixed (e.g., $p = p_0$ and $q = q_0$). Then the condition $T(p, q, r) \geq 0$ defines some union of segments and closed rays.

# 3 $pqr$-lemmas

A triple $(p, q, r)$ is called *acceptable* if $p$, $q$, $r \geqslant 0$ and $T(p, q, r) \geqslant 0$, i.e., if the non-negativity lemma applies. In other words, polynomial $x^3 - px^2 + qx - r = 0$ has three (perhaps multiple) real non-negative roots.

**24.** $r$**-lemma.** Fix some values $p = p_0$ and $q = q_0$ such that there exists at least one value of $r$ for which the triple $(p_0, q_0, r)$ is acceptable. Prove that such triple with the minimal value of $r$ corresponds to a triple $(a, b, c)$ in which either two numbers are equal, or $abc = 0$. Prove moreover that such triple with the maximal value of $r$ corresponds to a triple $(a, b, c)$ containing two equal numbers.

**25.** $q$**-lemma.** Fix some values $p = p_0$ and $r = r_0$ such that there exists at least one value of $q$ for which the triple $(p_0, q, r_0)$ is acceptable. Prove that such triples with minimal and maximal values of $q$ correspond to a triple $(a, b, c)$ containing two equal numbers.

**26.** $p$**-lemma.** Fix some values $q = q_0$ and $r = r_0 > 0$ such that there exists at least one value of $p$ for which the triple $(p, q_0, r_0)$ is acceptable. Prove that such triples with minimal and maximal values of $p$ correspond to a triple $(a, b, c)$ containing two equal numbers.
Does the same statement holds true when $r_0 = 0$?

Let us illustrate the use of the $pqr$-method by the following simple problem.

**Example.** Let $a, b, c$ be non-negative real numbers. Prove that

$$a^2 + b^2 + c^2 \geqslant ab + bc + ca.$$

**Proof.** The inequality can be rewritten as $p^2 - 2q \geqslant q \Leftrightarrow p^2 \geqslant 3q$. Fix $p = p_0$ and $r = r_0$. We need to prove that $q \leqslant \frac{p_0^2}{3}$. In other words, we shall show that $q$ is bounded from above by some constant. Notice that if the last inequality holds for greatest $q$, then it holds for all $q$. So it suffices to check the inequality for the largest value of $q$. It follows from the $q$-lemma that $q$ attains the maximal value when two of $a$, $b$, $c$ are equal. Assume without loss of generality that $a = b = x$, $c = z$. The inequality $q \leqslant \frac{p_0^2}{3}$ is equivalent to the desired one. Now, plugging $a = b = x$, $c = z$ into $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ we obtain

$$2x^2 + z^2 \geqslant x^2 + 2xz \Leftrightarrow (x - z)^2 \geqslant 0.$$

This argument shows that $a^2 + b^2 + c^2 \geqslant ab + bc + ca$ holds for all non-negative $a, b, c$. Equality holds if and only if $x = z$, i.e. $a = b = c$.

You can now return to the first section and use $pqr$-method in order to solve the problems 1-5.

# 4 Inequalities

*Suppose we want to check whether a symmetric inequality in 3 non-negative variables $a, b, c$ holds. Fixing two of the three variables $p$, $q$, and $r$, we can rewrite the inequality as $f \geqslant 0$, where $f$ is a function of the remaining variable. If $f$ is either monotonic or concave, then we need to check the inequality only in the case when $a = b$, and also when $a = 0$ if the non-fixed variable is $p$ or $q$.*

**For all inequalities you must find values of the numbers $a$, $b$, $c$ such that an inequality turns out to be equality!**

**27.** Let $a$, $b$, $c$ be non-negative real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that

$$(a-1)(b-1)(c-1) \geqslant 8.$$

**28.** Let $a$, $b$, $c$ be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{9-ab} + \frac{1}{9-bc} + \frac{1}{9-ca} \leqslant \frac{3}{8}.$$

**29.** Let $a$, $b$, $c$ be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geqslant \frac{2}{1+abc}.$$

**30.** Let $a$, $b$, $c$ be non-negative real numbers such that $a+b+c = 4$ and $a^2+b^2+c^2 = 6$. Prove that

$$a^6 + b^6 + c^6 \leqslant a^5 + b^5 + c^5 + 32.$$

**31.** Let $a$, $b$, $c$ be non-negative real numbers such that $ab + bc + ca \neq 0$. Prove that

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geqslant \frac{10}{(a+b+c)^2}.$$

**32.** Let $a$, $b$, $c$ be non-negative real numbers. Prove that

$$a^5 + b^5 + c^5 + abc(ab + bc + ca) \geqslant a^2b^2(a+b) + b^2c^2(b+c) + c^2a^2(c+a).$$

**33. a)** Let $a$, $b$, $c$ be non-negative real numbers. Prove that

$$\frac{a^4+b^4+c^4}{ab+bc+ca} + \frac{3abc}{a+b+c} \geqslant \frac{2}{3}(a^2+b^2+c^2).$$

**b)** Find the least non-negative real $k$ such that the inequality

$$k\frac{a^4+b^4+c^4}{ab+bc+ca} + (1-k)\frac{3abc}{a+b+c} \geqslant \frac{a^2+b^2+c^2}{3}$$

holds for all non-negative numbers $a$, $b$, $c$.

**34.** For positive real numbers $a$, $b$, $c$ define $X = \frac{a^2+b^2}{2c^2} + \frac{b^2+c^2}{2a^2} + \frac{c^2+a^2}{2b^2}$, $Y = \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}$. Prove that

$$4X + 69 \geqslant 27Y.$$

**35. a)** Let $P(a, b, c)$ be a homogeneous symmetric polynomial of degree not greater than 8. Find an algorithm checking whether $P$ is non-negative when $a, b, c$ are non-negative. We assume that we are ale to find extrema and zeroes of an arbitrary function in one variable.

**b)** Find an analogous algorithm for a homogeneous symmetric polynomial of degree not greater than 17.

**c*)** Find an analogous algorithm for any homogeneous symmetric polynomial.

In order to simplify the application of *pqr*-method, it is often useful to make a change of variables.

**36.** Let $a$, $b$, $c$ be non-negative real numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{c} = 3$. Prove that

$$2(a+b+c-2)^2 + (ab+bc+ca)(2+3(a+b+c)) \geqslant 35.$$

**37.** Let $a$, $b$, $c$ be real numbers such that $a, b, c \geqslant 1$ and $a + b + c = 9$. Prove that

$$\sqrt{ab+bc+ca} \leqslant \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

**38.** Let $a$, $b$, $c$ be positive real numbers. Prove that

$$\left(\frac{a}{b+c}\right)^3 + \left(\frac{b}{c+a}\right)^3 + \left(\frac{c}{a+b}\right)^3 + \frac{13abc}{(a+b)(b+c)(c+a)} \geqslant 2.$$

**39.** Let $a$, $b$, $c$ be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$\frac{a^2}{4-bc} + \frac{b^2}{4-ca} + \frac{c^2}{4-ab} \leqslant 1.$$

**40.** Let $a$, $b$, $c$ be real numbers. Prove that

$$(a^2+b^2+c^2)^2 \geqslant 3(a^3b + b^3c + c^3a).$$

Equality holds if either $a = b = c$ or

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}, \quad \frac{b}{\sin^2 \frac{4\pi}{7}} = \frac{c}{\sin^2 \frac{2\pi}{7}} = \frac{a}{\sin^2 \frac{\pi}{7}},$$

$$\frac{c}{\sin^2 \frac{4\pi}{7}} = \frac{a}{\sin^2 \frac{2\pi}{7}} = \frac{b}{\sin^2 \frac{\pi}{7}}.$$

*Hint.* Perform the substitution $a = x + 2ty$, $b = y + 2tz$, $c = z + 2tx$ for $t \in \mathbb{R}$.

# 5 Unusual conditions

Up to this point we have considered rather simple conditions on $a$, $b$, $c$, e.g. $a$, $b$, $c \geqslant 0$ or $a+b+c=3$. These conditions can be easily written in terms of $p$, $q$, $r$. Is it possible to formulate unusual conditions in terms of $p, q, r$?

**41.** Find conditions on numbers $p$, $q$, $r$ necessary and sufficient for
**a)** numbers $a$, $b$, $c$ to be not less than 1;
**b)** numbers $a$, $b$, $c$ to be side lengths of a triangle (perhaps degenerate);
**c)** non-negative real numbers $a$, $b$, $c$ to satisfy $2\min(a,b,c) \geqslant \max(a,b,c)$.

**42.** Let $a$, $b$, $c$ be real numbers such that $a$, $b$, $c \in \left[\frac{1}{3}, 3\right]$. Prove that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geqslant \frac{7}{5}.$$

**43.** Let $a$, $b$, $c$ be side lengths of a triangle. Prove that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geqslant 6\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a}\right).$$

**44.** Prove that there exists a polynomial $S(x,y,z)$ such that the following conditions are equivalent: **(i)** $a, b, c$ are real numbers; **(ii)** $S(x,y,z) \geqslant 0$, where $x = a+b+c$, $y = a^2+b^2+c^2$, $z = a^3+b^3+c^3$.

**45.** Prove that the following conditions are equivalent: **(i)** $s \in \left[\frac{86}{9}, 10\right]$; **(ii)** there exist real numbers $a, b, c$ such that $a+b+c=4, a^2+b^2+c^2=6, a^3+b^3+c^3=s$.

**46.** (USA TST, 2001) Let $a$, $b$, $c$ be non-negative real numbers such that $a^2+b^2+c^2+abc=4$. Prove that

$$ab+bc+ca-abc \leqslant 2.$$

**47.** (China-West, 2004) Let $a$, $b$, $c$ be positive real numbers. Prove that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \leqslant \frac{3}{2}\sqrt{2}.$$

**48.** Let $a$, $b$, $c$ be non-negative real numbers such that $a^2+b^2+c^2+nabc=n+3$ for some real $n$.

**a)** Assume that $0 \leqslant n \leqslant \frac{3}{2}$. Prove that $a+b+c \leqslant 3$.

**b)** Assume that $\frac{3}{2} \leqslant n \leqslant 2$. Prove that $a+b+c \leqslant \sqrt{2(n+3)}$.

**c)** Assume that $n=2$. Prove that $ab+bc+ac-abc \leqslant \frac{5}{2}$.

**49.** Let $a$, $b$, $c$ be non-negative numbers such that $a^2+b^2+c^2+abc=4$. Prove that

$$a^2b^2+b^2c^2+c^2a^2 \geqslant abc(2+\sqrt{4-3abc}).$$

# 6 Special cases

In this section we consider complicated symmetric conditions containing all three elementary symmetric polynomials. If a symmetric inequality (on non-negative numbers) obey conditions of this type, then the $pqr$-method can not be applied. Indeed, if we fix two numbers of the triple $p$, $q$, $r$, then the third one will almost certainly be uniquely defined. What is to be done?

Fix one number of the numbers $p$, $q$, $r$ (to be precise, $r$). Then our condition will describe the relationship between the remaining two numbers. If we express one of them by the other, then our condition states that the first elementary symmetric polynomial (to be precise, $p$) is a function of the second one (to be precise, $q$). Thus, the triple $(p, q, r)$ is uniquely determined by $q$. This means that our inequality can be naturally written as $h(q) \geqslant 0$, where $h$ is a function of real non-negative variable. If $h$ is monotonic or concave, then it suffices to prove our inequality only for extremal $q$. So, the problem is reduced to the problem of finding extremal $q$ (when condition and $r$ are fixed).

Let us spend some time setting the scene.

Let $\mathbb{R}^n$ denote a set of ordered groups of $n$ real numbers. For small $n$ the set $\mathbb{R}^n$ has a natural geometric interpretation. Namely, $\mathbb{R}^1$ is a line, $\mathbb{R}^2$ is a plane. Hence we say that the elements of $\mathbb{R}^n$ are points.

The *distance* between two points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is defined by $||x - y|| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$. In particular, for $n = 1$ distance is equal to the modular of the difference between corresponding numbers, for $n = 2$ distance is the well-known distance between points.

Let $M$ be a subset of $\mathbb{R}^n$. The function $f : M \to \mathbb{R}$ is *continuous* in the point $x \in M$ if for any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon)$ such that if for $y \in \mathbb{R}^n$ holds $||x - y|| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

**Example.** Let us prove that the function $f(x) = x^2$ is continuous for any $x \in \mathbb{R}$. Note that

$$|x^2 - y^2| = |x - y| \cdot |(y - x) + 2x| \leqslant |x - y| \cdot (|x - y| + |2x|) < \delta^2 + 2|x|\delta.$$

Suppose $\delta = \min(\sqrt{\frac{\varepsilon}{2}}, \frac{\varepsilon}{4|x|})$. Since both items are not greater than $\frac{\varepsilon}{2}$, it follows that $|x^2 - y^2| < \varepsilon$.

You may use results of the next three problems in the sequel without proof.

**50.** Prove that the function $f(x) = \frac{1}{x}$ is continuous for any $x \in (0, +\infty)$, the function $f(x) = \sqrt{x}$ is continuous for any $x \in (0, +\infty)$, the function $f(x, y) = x + y$ is continuous for any $(x, y) \in \mathbb{R}^2$.

**51.** Let the function $g(x)$ be continuous for any $x \in M \subset \mathbb{R}^n$, let range of $g(x)$ belongs to $N \subset \mathbb{R}$. Let the real-valued function $f(y)$ be continuous for any $y \in N$. Prove that the function $f(g(x))$ is continuous for any $x \in M$.

**52. a)** Prove that the polynomial $P(x)$ in one variable is continuous for any $x \in \mathbb{R}$.

**b)** Prove that the polynomial $P(x_1, \ldots, x_n)$ in $n$ variables is continuous for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

We say that the real numbers $a$, $b$, $c$ obey the *symmetric relation* $G$, if the equality $g(p, q, r) = 0$ holds, where $g$ is a continuous function in 3 variables.

**53.** Let the variables $a$, $b$, $c$ obey the symmetric condition $G$. Fix some value $r \geqslant 0$ such that there exist $p$, $q \geqslant 0$ for which the triple $(p, q, r)$ is acceptable and $G(p, q, r) = 0$. Assume that the condition $G$ is equivalent to $p = f(q)$ (while $r$ is fixed), where $f$ is a function defined below, and a set of admissible $q$ is bounded. Prove that $q$ attains its maximal and minimal value. Moreover, in both cases two variables of $a$, $b$, $c$ are equal when

**a)** $f$ is a linear function;

**b)** $f$ is a polynomial.

**54.** Let the variables $a$, $b$, $c$ obey the symmetric condition $G$. Let $(x, y, z)$ be an arbitrary permutation of $(p, q, r)$. Fix some value $z \geqslant 0$ such that there exist $x, y \geqslant 0$ for which the triple $(p, q, r)$ is acceptable and $G(p, q, r) = 0$. Assume that the condition $G$ is equivalent to $x = f(y)$ (while $z$ is fixed), where $f$ is a polynomial, and a set of admissible $y$ is bounded. Prove that $y$ attains its maximal and minimal values. Moreover, if $z = r$, then the equality $(a - b)(b - c)(c - a) = 0$ holds for any extreme point, otherwise $abc(a - b)(b - c)(c - a) = 0$ holds for any extreme point.

**Remark.** Let $I$ be a union of segment containing admissible $y$. Statement of the previous problem is also true when $f$ is continuous function for any point of $I$. If you solve the previous problem, you may use this fact in the sequel without proof.

**Example.** Let $a$, $b$, $c$ be non-negative numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$a + b + c \geqslant 2 + \sqrt{abc(4 - a - b - c)}.$$

**Proof.** The desired inequality can be written as

$$p \geqslant 2 + \sqrt{r(4 - p)}, \quad (*)$$

while the relation $a^2 + b^2 + c^2 + abc = 4$ can be written as $\frac{p^2 - 4 + r}{2} = q$. If variable $r$ is fixed, we can write the desired inequality as $f(p) \geqslant 0$, where $f(x) = x - 2 - \sqrt{r(4 - x)}$ is a monotonic function. Therefore, it suffices to check $(*)$ for the smallest value of $p$. From the problem 54 it follows that it suffices to check the desired inequality for $a = b$. If $a = b$, then $c = 2 - a^2$. The desired inequality can be written as

$$a^4(a - 1)^2 \geqslant 0.$$

Equality holds if $a=b=c=1$; $a=b=0, c=2$; $a=c=0, b=2$; $b=c=0, a=2$.

**55.** Let $a$, $b$, $c$ be non-negative real numbers such that $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=10$. Prove that

$$\frac{9}{8} \leqslant \frac{a^2+b^2+c^2}{ab+bc+ca} \leqslant \frac{6}{5}.$$

**56.** Let $a$, $b$, $c$ be positive real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Prove that

$$(a+b+c)(1+abc) \geqslant 6.$$

**57.** Let $a$, $b$, $c$ be non-negative real numbers such that $ab+bc+ca=a^3+b^3+c^3$. Prove that

$$ab+bc+ca \geqslant a^2b^2+b^2c^2+c^2a^2.$$

**58.** Let $a, b, c$ be non-negative real numbers, $p, q, r$ are defined naturally. Prove that if $q+r=4$, then

$$p^3-27r \geqslant 7(p^2-3q).$$

**59.** Let $a$, $b$, $c$ be positive real numbers such that $a^2+b^2+c^2=ab+bc+ca+(abc-1)^2$. Prove that

$$ab+bc+ca+3 \geqslant 2(a+b+c).$$

# 7 $n$ variables

Up to this point we considered symmetric inequalities in 3 variables. What can we say if there are more than 3 variables?

The *ball* (the *sphere*) in $\mathbb{R}^n$ of radius $r$ and center $x$ is a set of points $y$ such that $||x-y|| \leqslant r$ ($||x-y||=r$, respectively). In particular, if $n=2$, then ball is a circle and sphere is a circumference.

Consider a sequence $z_1, z_2, \ldots$ in $\mathbb{R}^n$. We say that this sequence *converges* to the point $z$, if for any $\varepsilon>0$ the exterior of the ball of radius $\varepsilon$ and center $z$ contains only finitely many elements of this sequence.

**Example.** Since there are only finitely many elements of the sequence $z_n=\frac{1}{n}$, where $n \in \mathbb{N}$, in the exterior of the segment $[-\varepsilon, \varepsilon]$, it follows that $z_n$ converges to $z=0$.

Consider a set of points $M$ in $\mathbb{R}^n$. The point $z$ is called *a limit point* of $M$ if there exists a sequence of points $z_1, z_2, \ldots$, where $z_i \in M$ and $z_i \neq z$, converging to $z$. We say that the set $M$ in $\mathbb{R}^n$ is *closed*, if it contains all its limit points.

**Example.** Since 0 is a limit point of the interval $(0,1)$ but does not belong to it, this interval is not a closed set. Conversely, the segment $[0,1]$ is a closed set.

We say that the set $M$ in $\mathbb{R}^n$ is *bounded*, if there exists constant $C$ such that for any $x = (x_1, x_2, \dots, x_n) \in M$ and for any $i$, $1 \leqslant i \leqslant n$, $|x_i| < C$. The set $M$ in $\mathbb{R}^n$ is called *a compact set* if $M$ is closed and bounded.

**60.** Is the segment $[0, 1]$ a compact set? The ray $[0, +\infty)$? The point $(1, 1, \dots, 1)$? The ball of radius 1 and center $(1, 1, \dots, 1)$? The same ball without the point $(1, 1, \dots, 1, 0)$? The sphere of radius 1 and center $(1, 1, \dots, 1)$?

You may use result of the next problem in the sequel without proof.

**61.** Consider a number of relations

$$P_1(x_1, \dots, x_n) = 0, \dots, P_s(x_1, \dots, x_n) = 0, P_{s+1}(x_1, \dots, x_n) \geqslant 0, \dots, P_m(x_1, \dots, x_n) \geqslant 0,$$

where $m \in \mathbb{N}$, $s \in \mathbb{N}_0$, $P_1, \dots, P_m$ are polynomials in $n$ variables. Let $M$ be a set of points obeying this relations. Prove that if $M$ is bounded, then $M$ is a compact set.

We need the following classical theorem of calculus. You can use it without proof.

**The Weierstrass theorem.** Continuous function on a compact set attains its maximum and minimum.

**Example.** The function $x^2$ on the segment $[0, 1]$ attains its maximum and minimum. However the Weierstrass theorem doesn't imply an analogous statement for $x^2$ on the half-interval $(0, 1]$.

**62.** Let $f(x_1, x_2, \dots, x_n)$ be a symmetric continuous function on the compact set $M \subset \mathbb{R}^n$. Let $f$ be such that if $a_4, \dots, a_n$ are fixed (and there exist $a_1$, $a_2$, $a_3$ such that $(a_1, \dots a_n) \in M$), then the function

$$h(x_1, x_2, x_3) = f(x_1, x_2, x_3, a_4, \dots, a_n)$$

attains the maximal and minimal values only when $(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = 0$ (or $x_1 x_2 x_3 (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = 0$). Prove that the function $f$ attains its maximum and minimum. Moreover, in both cases there are less than 3 different numbers among $x_1, \dots, x_n$ (or there are less than 3 different non-zero numbers among $x_1, \dots, x_n$).

**Example.** Let $a$, $b$, $c$, $d$ be non-negative real numbers such that $a + b + c + d = 4$. Prove that

$$3(a^2 + b^2 + c^2 + d^2) + 4abcd \geqslant 16.$$

**Proof.** From the problem 61 it follows that the given relations define a compact set. By definition, put

$$f(a, b, c, d) = 3(a^2 + b^2 + c^2 + d^2) + 4abcd.$$

From the problem 52 we can say that $f$ is a continuous function. Fix $d = d_0 \geqslant 0$. By definition, put $p = a + b + c$, $q = ab + bc + ac$, $r = abc$. Consider the function

$$h(a, b, c) = f(a, b, c, d_0) = 3(a^2 + b^2 + c^2 + d_0^2) + 4abcd_0 = 3p^2 - 6q + 4d_0 r + 3d_0^2 =$$

$$= 3p^2 - 6q + 4d_0r + 3d_0^2.$$

Fix $p$ and $r$. Since $h$ linear in $q$, it follows that $h$ attains its minimum only when $(a-b)(b-c)(c-a) = 0$. From the problem 62 it follows that it suffices to prove the desired inequality when there are less than 3 different numbers among $a$, $b$, $c$, $d$.

- Considering $a = b$ and $c = d = 2 - a$, we obtain

$$4(a-1)^2((a-1)^2 + 1) \geqslant 0.$$

  Equality holds if $a = b = c = d = 1$.

- Considering $a = b = c$ and $d = 4 - 3a$, we obtain

$$4(a-1)^2(4-3a)(a+2) \geqslant 0.$$

  Since $a \leqslant \frac{4}{3}$, if follows that inequality holds true. Equality holds if $a = b = c = \frac{4}{3}, d = 0$; $a = b = d = \frac{4}{3}, c = 0$; $a = c = d = \frac{4}{3}, b = 0$; $b = c = d = \frac{4}{3}, a = 0$.

**63.** Let $a$, $b$, $c$, $d$ be non-negative real numbers such that $a + b + c + d = 1$. Prove that

$$(1-a^2)^2 + (1-b^2)^2 + (1-c^2)^2 + (1-d^2)^2 \geqslant 3.$$

**64.** (Russia TST, 2015) Let $a$, $b$, $c$, $d$ be non-negative real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Prove that

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \leqslant 1.$$

**65.** (IMO Shortlist, 2010) Let $a$, $b$, $c$, $d$ be real numbers such that $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

**a)** $abcd \leqslant 3$;

**b)** $36 \leqslant 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4) \leqslant 48$.

**66.** Let $a$, $b$, $c$, $d$ be non-negative real numbers. Prove that

$$a^3 + b^3 + c^3 + d^3 + 4\sqrt[4]{a^3b^3c^3d^3} \geqslant 2(abc + bcd + cda + dab).$$

**67.** (Russia, 2016) Let $a$, $b$, $c$, $d$ be positive real numbers such that $a + b + c + d = 3$.

**a)** Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \leqslant \frac{1}{a^2b^2c^2d^2}.$$

**b)** Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leqslant \frac{1}{a^3b^3c^3d^3}.$$

**c)** Let $x$ be real number not less than 2. Prove that

$$\frac{1}{a^x} + \frac{1}{b^x} + \frac{1}{c^x} + \frac{1}{d^x} + \left|\frac{(1-\frac{1}{a})(1-\frac{1}{b})(1-\frac{1}{c})(1-\frac{1}{d})}{2}\right|^x \leqslant \frac{1}{a^xb^xc^xd^x}.$$

**68.** Let $a$, $b$, $c$, $d$ be non-negative numbers such that $a+b+c+d=4$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+4 \geqslant 2(a^2+b^2+c^2+d^2).$$

**69.** Let $a$, $b$, $c$, $d$, $e$ be real numbers such that $a+b+c+d+e=20$, $a^2+b^2+c^2+d^2+e^2=100$. Find extrema of

$$abcd+abce+abde+acde+bcde.$$

**70.** Let $a$, $b$, $c$, $d$ be positive real numbers such that $a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$. Prove that

$$(a+b+c+d)(17+46abcd) \geqslant 252.$$

# 8 Additional problems

In the problems below you can apply *pqr*-method in the non-obvious way.

**71.** (APMO, 2004) Let $a$, $b$, $c$ be real numbers. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geqslant 9(ab+bc+ca).$$

**72.** (Iran, 2005) Let $a$, $b$, $c$ be real non-negative numbers such that $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$. Prove that

$$ab+bc+ca \leqslant \frac{3}{2}.$$

**73.** Let $a$, $b$, $c$ be side lengths of an acute triangle and $R$ be a circumradius of this triangle. Prove that

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} > 5R.$$

**74.** (Shortlist IMO, 2011) Let $a$, $b$, $c$ be side lengths of a triangle such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{(b+c-a)^2}+\frac{b}{(a+c-b)^2}+\frac{c}{(a+b-c)^2} \geqslant \frac{3}{(abc)^2}.$$

**75.** The polynomial $G(a,b,c)$ in 3 variables is called *cyclic* if $G(a,b,c)=G(b,c,a)=G(c,a,b)$. Prove that any cyclic polynomial in $a$, $b$, $c$ can be written as $X(a,b,c)+Y(a,b,c)(l_{12}-l_{21})$ ($X$ and $Y$ are symmetric polynomials).

**76.** Let $a$, $b$, $c$ be positive real numbers such that $a^2+b^2+c^2=\frac{5}{2}(ab+bc+ca)$. Prove that

$$(a^2+b^2+c^2)^2 \geqslant \frac{25}{8}(a^3b+b^3c+c^3a).$$

**77.** Let $a$, $b$, $c$ be non-negative real numbers such that $a+b+c=3$. Prove that

$$(a^2b+b^2c+c^2a)(ab+bc+ca) \leqslant 9.$$

**78.** Let $a$, $b$, $c$ be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a^2+3b^2}{a+3b}+\frac{b^2+3c^2}{b+3c}+\frac{c^2+3a^2}{c+3a} \geqslant 3.$$

**79.** Let $a$, $b$, $c$ be non-negative real numbers. Prove that

$$10(a+b+c)^5 \geqslant 81(a^2+b^2+c^2)(a^2b+b^2c+c^2a+7abc).$$

**80.** (Korea, 2012) Let $a$, $b$, $c$ be non-negative real numbers such that $a^2+b^2+c^2=2abc+1$. Find the maximum of

$$(a-2bc)(b-2ca)(c-2ab).$$

**81.** Let $a$, $b$, $c$ be real numbers such that $a$, $b$, $c \in [0,1]$ and $(1-a)(1-b)(1-c)=abc$. Prove that

$$a^2+b^2+c^2+\frac{a+b+c}{2} \geqslant \frac{3}{2}.$$

**82.** Let $a$, $b$, $c$ be non-negative real numbers such that $a^2+b^2+c^2+abc=4$ and $a+b+c \geqslant \sqrt{8}$. Prove that

$$a+b+c \geqslant 2+\sqrt[3]{abc}.$$

**83.** Let $a$, $b$, $c$ be non-negative real numbers such that $a^2+b^2+c^2-(ab+bc+ca)=11(a+b+c-3)$. Prove that

$$\sqrt{a+2}+\sqrt{b+2}+\sqrt{c+2} \geqslant \sqrt{a+b+c+24}.$$

**84.** Let $a$, $b$, $c$ be positive real variables such that $abc=1$. Prove that

**a)**

$$a^{10}+b^{10}+c^{10} \geqslant 3+45\left((a-1)^2+(b-1)^2+(c-1)^2\right);$$

**b*)**

$$a^{10}+b^{10}+c^{10} \geqslant 3+46\left((a-1)^2+(b-1)^2+(c-1)^2\right);$$

**c)** there exists $k>0$ such that for any natural number $n \geqslant 3$

$$a^n+b^n+c^n \geqslant 3+kn((a-1)^2+(b-1)^2+(c-1)^2);$$

**d*)** there exists $k>0$ such that for any natural number $n \geqslant 3$

$$a^n+b^n+c^n \geqslant 3+kn^2((a-1)^2+(b-1)^2+(c-1)^2),$$

but there is no $\varepsilon > 0$ such that there exists $k' > 0$ such that for any natural number $n \geqslant 3$

$$a^n + b^n + c^n \geqslant 3 + k' n^{2+\varepsilon}((a-1)^2 + (b-1)^2 + (c-1)^2).$$

**85.** Let $a$, $b$, $c$ be non-negative real numbers such that $(a+1)(b+1)(c+1) = 84$, $(a+b+c)(ab+bc+ac) = \frac{14256}{abc}$. Find the maximum of

$$(a+b+c)^2 + (ab+bc+ac)^2 + a^2b^2c^2.$$

**86.** Let $a$, $b$, $c$, $d$ be non-negative real numbers such that $a+b+c+d = 3$, $a^2+b^2+c^2+d^2 = 5$, $a^4+b^4+c^4+d^4 = 17$. Find the maximum of $d$.

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