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Almost all inequalities considered in our project are symmetric. Hence if plugging (a_0, b_0, c_0) into our inequality we obtain an equality, then plugging a permutation of (a_0, b_0, c_0) into the same inequality we also obtain an equality. For brevity, we shall mention only triples that are not permutations of one another.

Therefore after reducing our inequality we assume without loss of generality that either $a = b$ or $a = 0$.

1. Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that

$$1 + 12abc \geqslant 4(ab + bc + ca).$$

Solution. The desired inequality can be written as $1 + 12r \geqslant 4q$, and the relation $a + b + c = 1$ can be written as $p = 1$. Fix p and r. Then q attains the maximal value when $a = b$ and $c = 1 - 2a$. Substituting $a = b$ and $c = 1 - 2a$, we get

$$24a^3 - 24a^2 + 8a - 1 \leqslant 0 \Leftrightarrow 24\left(a - \frac{1}{3}\right)^3 - \frac{1}{9} \leqslant 0.$$

The left hand side attains its maximal value when a does, i.e. $a = \frac{1}{2}$. Equality is attained when $a = b = \frac{1}{2}$, $c = 0$.

2. Let a, b, c be real numbers such that

$$a + b + c = 9, \quad ab + bc + ca = 24.$$

Prove that $16 \leqslant abc \leqslant 20$. Prove moreover that for any $r \in [16, 20]$ there exist real numbers a, b, c such that $a + b + c = 9$, $ab + bc + ca = 24$, $abc = r$.

Solution. From the problem 22 it follows that numbers a, b, c are real if and only if p, q, r are real and $T(p, q, r) \geqslant 0$. Substituting $p = 9$ and $q = 24$, we get

$$T(9, 24, r) = -27(r - 16)(r - 20).$$

Its value is non-negative if and only if $r \in [16, 20]$.

3. Let P be a symmetric polynomial of degree not greater than 5. Prove that if $P(a, a, c) \geqslant 0$ and $P(0, b, c) \geqslant 0$ for all non-negative real numbers a, b, c, then $P(a, b, c) \geqslant 0$ for all non-negative real numbers a, b, c.

Solution. We have $P(a, b, c) = A(p, q)r + B(p, q)$. Fix p and q. Since $P(a, b, c)$ is linear in r, it follows that $P(a,b,c)$ attains its maximal and minimal values only when either $a=b$ or $a=0$.

4. (Russia TST, 2015) Let a, b, c be non-negative real numbers such that $1+a+b+c = 2abc$. Prove that

$$\frac{ab}{1+a+b}+\frac{bc}{1+b+c}+\frac{ca}{1+c+a}\geqslant\frac{3}{2}.$$

Solution. The desired inequality can be written as

$$2q^2-q(p+1)+9r+2pr-3(p+1)^2\geqslant 0,$$

and the relation $1+a+b+c = 2abc$ can be written as $1+p=2r$. Fix p and r. The left hand side is a quadratic polynomial in q, its leading coefficient is positive. Its minimal value is attained when $q_0=\frac{p+1}{4}$.

Let us prove an inequality $4q-(p+1)\geqslant 0$. Fix p and r. Left hand side attains its minimal value when q does. Substituting $a=b$, $c=\frac{1+2a}{2a^2-1}$ in $4q-(p+1)\geqslant 0$, we get

$$8a^4-4a^3+10a^2+8a\geqslant 0,$$

which is correct.

Since $4q-(p+1)\geqslant 0$, it follows that left hand side is an increasing function. Substituting $a=b$ and $c=\frac{1+2a}{2a^2-1}$ in the desired inequality, we get

$$a(2a^2-2a-1)^2\geqslant 0.$$

Equality is attained when $a=b=c=\frac{1+\sqrt{3}}{2}$.

5. (Iran TST, 1996) Let a, b, c be non-negative real numbers such that $ab+bc+ca\neq 0$. Prove that

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\geqslant\frac{9}{4}.$$

Solution. The desired inequality can be written as

$$\frac{4qpr+q(p^4-2p^2q+q^2)}{(pq-r)^2}\geqslant\frac{9}{4}$$

Fix p and q. Since $r\leqslant pq$, it follows that left hand side attains its minimal value when r attains either maximal or minimal value.

- If $a=0$, then the desired inequality can be written as

  $$\frac{(b-c)^2(b^2+bc+c^2)}{2bc(b+c)^2}\geqslant 0.$$

  Equality is attained when $a=0$, $b=c$.

- If $a = b$, then the desired inequality can be written as

$$t(t-1)^2 \geqslant 0,$$

where $t = \frac{c}{b}$. Equality is attained when $a = b = c$.

6. Express a^2+b^2, a^3+b^3, a^4+b^4, $(a-b)^2$ in terms of $a+b$ and ab.

Solution.

$$a^2+b^2 = (a+b)^2 - 2ab,$$
$$a^3+b^3 = (a+b)(a^2-ab+b^2) = (a+b)((a+b)^2-3ab),$$
$$a^4+b^4 = (a^2+b^2)^2 - 2a^2b^2 = ((a+b)^2-2ab)^2 - 2(ab)^2,$$
$$(a-b)^2 = a^2+b^2-2ab = (a+b)^2 - 4ab.$$

7. Express polynomials $a^2+b^2+c^2$, $a^2b+a^2c+b^2a+b^2c+c^2a+c^2b$, $a^3+b^3+c^3$, $(ab)^2+(bc)^2+(ca)^2$, $a^4+b^4+c^4$, $(a+b)(b+c)(c+a)$ in terms of p, q, r.

Solution.

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = p^2-2q,$$
$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b = (a+b+c)(ab+ac+bc) - 3abc = pq-3r,$$
$$a^3+b^3+c^3 = p^3 - 3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b) - 6abc = p^3-3pq+3r,$$
$$(ab)^2+(bc)^2+(ca)^2 = (ab+bc+ca)^2 - 2(ab^2c+abc^2+a^2bc) = q^2-2pr,$$
$$a^4+b^4+c^4 = (a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2) = p^4-4p^2q+2q^2+4pr,$$
$$(a+b)(b+c)(c+a) = (p-c)(p-a)(p-b) = p^3 - p\cdot p^2 + pq - r = pq - r.$$

8. By definition, put $s_k = a^k+b^k+c^k$ for any non-negative integer number k. Express s_k $(k \geqslant 3)$ in terms of p, q, r, s_{k-1}, s_{k-2} and s_{k-3}.

Solution. Let us prove that $s_k = ps_{k-1} - qs_{k-2} + rs_{k-3}$.

$$ps_{k-1}-qs_{k-2}+rs_{k-3} = (a+b+c)(a^{k-1}+b^{k-1}+c^{k-1}) - (ab+bc+ca)(a^{k-2}+b^{k-2}+c^{k-2}) +$$
$$+ abc(a^{k-3}+b^{k-3}+c^{k-3}) = (s_k + ab^{k-1} + ac^{k-1} + ba^{k-1} + bc^{k-1} + ca^{k-1} + cb^{k-1}) -$$
$$- (ab^{k-1} + ac^{k-1} + ba^{k-1} + bc^{k-1} + ca^{k-1} + cb^{k-1} + abc^{k-2} + ab^{k-2}c + a^{k-2}bc) +$$
$$+ (abc^{k-2} + ab^{k-2}c + a^{k-2}bc) = s_k.$$

9. Prove that any symmetric polynomial in a, b, c can be expressed as a polynomial in p, q, r.

Solution. Let $G(a,b,c)$ be a given polynomial. $G = G_1+G_2+G_3$, where all monomials in G_i contain i variables, $i = 1, 2, 3$. From the previous problem it follows that G_1 can be expressed as a polynomial in p, q, r. Since an equality

$$s_k s_l - s_{k+l} = a^k b^l + a^k c^l + b^k a^l + b^k c^l + c^k a^l + c^k b^l$$

holds, it follows that G_2 can be expressed as a polynomial in p, q, r. Finally, for a sum $a^k b^l c^m + \ldots$ in G_3 we factorize $(abc)^n$ (where $n = \min(k, l, m)$) and reduce our problem to the previous cases.

10. Let t be a real number. Solve a system of equations: $\begin{cases} a+b+c=t, \\ a^2+b^2+c^2=t^2, \\ a^3+b^3+c^3=t^3. \end{cases}$

Solution. $\begin{cases} a+b+c=t \\ a^2+b^2+c^2=t^2 \\ a^3+b^3+c^3=t^3 \end{cases} \Leftrightarrow \begin{cases} p=t \\ p^2-2q=t^2 \\ p^3-3pq+3r=t^3 \end{cases} \Leftrightarrow \begin{cases} p=t \\ q=0 \\ r=0 \end{cases}$

If $q = 0$, then two of a, b, c are equal to zero. So, there are 3 solutions: $a = b = 0$, $c = t$ and its permutations.

11. Let a, b, c be real numbers. Prove that $q^2 \geqslant 3pr$.

Solution. Substituting a, b, c in the desired inequality, we get

$$(ab+bc+ca)^2 \geqslant 3(a+b+c)abc. \Leftrightarrow \frac{1}{2}(ab-ac)^2 + \frac{1}{2}(ab-bc)^2 + \frac{1}{2}(ac-bc)^2 \geqslant 0.$$

12. Let a, b, c be non-negative real numbers. Prove that $\frac{p}{3} \geqslant \sqrt{\frac{q}{3}} \geqslant \sqrt[3]{r}$.

Solution.

$$\frac{p}{3} \geqslant \sqrt{\frac{p}{3}} \Leftrightarrow p^2 \geqslant 3q \Leftrightarrow a^2+b^2+c^2 \geqslant ab+bc+ca.$$

$$\sqrt{\frac{q}{3}} \geqslant \sqrt[3]{r} \Leftrightarrow \frac{ab+bc+ca}{3} \geqslant \sqrt[3]{a^2b^2c^2}.$$

Using the AM-GM inequality for ab, bc, ca, we get the desired inequality.

13. Prove that a and b are roots of equation $x^2 - px + q = 0$ and there are no other roots.

Solution. $(x-a)(x-b) = x^2 - (a+b)x + ab = x^2 - px + q$. Left hand side equals 0 if and only if either $x = a$ or $x = b$.

14. Prove that if p and q are real, then either a and b are both real or b is the complex conjugate of a.

Solution. If a is real, then $b = p - a$ is also real as a difference of real numbers. Assume that $a = x + yi$, where $y \neq 0$. Then $b = p - a = (p - x) - yi$. Since

$$q = ab = (x+iy)(p-x-yi) = (xp - x^2 + y^2) + (py - 2xy)i,$$

it follows that $py - 2xy = 0$ and $p = 2x$, so $b = x - yi = \overline{a}$.

15. Prove that if p and q are real, then $(a - b)$ is either real or pure imaginary.

Solution. It is an obvious corollary of the previous problem.

16. Which conditions (particularly, inequalities) should satisfy p and q for a and b to be real?

Solution. p and q should be real numbers and $p^2 - 4q$ should be non-negative.

17. Prove that a and b are real and non-negative if and only if p and q are non-negative real numbers which satisfy conditions from the previous problem.

Solution. It is an obvious corollary of the previous problem.

18. Prove that a, b, c are roots of equation $x^3 - px^2 + qx - r = 0$ and there are no other roots.

Solution. The proof is similar to the proof of the problem 13.

19. Prove that for real numbers p', q', r' there exist complex numbers a', b', c' (unique up to a permutation) such that $p' = a' + b' + c'$, $q' = a'b' + b'c' + c'a'$, $r' = a'b'c'$. Prove moreover that either numbers a', b', c' are real or a' is real and b' is a complex conjugate of c' (up to a permutation).

Solution. Since any polynomial of odd degree has a real root, it follows that the desired statement is an obvious corollary of the problem 14.

20. Assume that p, q, and r are real numbers. Prove that if a, b, c are real, then $(a-b)(b-c)(c-a)$ is real, otherwise it is pure imaginary.

Solution. If a, b, c are real numbers, then the problem is obvious. Let a be a real number, $b = x + yi$, $c = x - yi$. Then

$$(a-b)(b-c)(c-a) = -(a^2 - (b+c)a + bc)(b-c) = -2yi(a^2 + 2ax + x^2 + y^2).$$

21. Prove that

$$(a-b)^2(b-c)^2(c-a)^2 = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2.$$

Solution.

$$(a-b)^2(b-c)^2(c-a)^2 = (ab^2+bc^2+ca^2-a^2b-b^2c-c^2a)^2 = (ab^2+bc^2+ca^2+a^2b+b^2c+c^2a)^2-$$

$$-4(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) = (pq-3r)^2 - 4((ab)^3+(bc)^3+(ca)^3)-$$

$$-4(a^4bc+ab^4c+abc^4+3(abc)^2) = (pq-3r)^2-4(q^3-3pqr+3r^2)-4r(p^3-3pq+3r)-12r^2 =$$

$$= -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2.$$

22. Criterion for reality. Let (p,q,r) be a triple of real numbers. Prove that the numbers a, b, c (which are defined as the roots of $x^3 - px^2 + qx - r = 0$ counting multiplicities) are real if and only if $T(p,q,r) \geqslant 0$, where $T(p,q,r)$ is a polynomial in 3 variables defined as $T(p,q,r) = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2$.

Solution. It is an obvious corollary of the problems 20 and 21.

23. Non-negativity lemma. Prove that p, q, $r \geqslant 0$ and $T(p,q,r) \geqslant 0$ if and only if a, b, c are non-negative real numbers.

Solution. It is an obvious corollary of the previous problem.

24. r**-lemma.** Fix some values $p = p_0$ and $q = q_0$ such that there exists at least one value of r for which the triple (p_0, q_0, r) is acceptable. Prove that such triple with the minimal value of r corresponds to a triple (a, b, c) in which either two numbers are equal, or $abc = 0$. Prove moreover that such triple with the maximal value of r corresponds to a triple (a, b, c) containing two equal numbers.

Solution. For triple (p, q, r) to be acceptable it is necessary to have $T(p_0, q_0, r) \geqslant 0$ and $r \geqslant 0$ (p_0, q_0 are non-negative, by the condition). Since $T(p_0, q_0, r)$ is a quadratic function in r with non-negative leading coefficient, it follows that $T(p_0, q_0, r) \geqslant 0$ defines a segment. Therefore the maximal value of r is right end-point of that segment and the minimal one is either left end-point or 0. Since $T(p_0, q_0, r) = 0$ in the end-points, it follows that two numbers of a, b, c are equal ($T(p, q, r) = (a-b)^2(b-c)^2(c-a)^2$). If $r = 0$, then one number of a, b, c equals 0.

25. q**-lemma.** Fix some values $p = p_0$ and $r = r_0$ such that there exists at least one value of q for which the triple (p_0, q, r_0) is acceptable. Prove that such triples with minimal and maximal values of q correspond to a triple (a, b, c) containing two equal numbers.

Solution. $T(p_0, q, r_0)$ is a third degree polynomial in q with negative leading coefficient. Constant term equals $-4p_0^3 r_0 - 27r_0^2$, so it is negative. Therefore $T(p_0, q, r_0) \geqslant 0$ defines a union of a segment with non-negative end-points and a ray $(-\infty, q']$, where $q' \leqslant 0$. Now the proof is trivial.

26. p**-lemma.** Fix some values $q = q_0$ and $r = r_0 > 0$ such that there exists at least one value of p for which the triple (p, q_0, r_0) is acceptable. Prove that such triples with minimal and maximal values of p correspond to a triple (a, b, c) containing two equal numbers.
Does the same statement holds true when $r_0 = 0$?

Solution. The proof is similar to the proof of the problem 25 when $r_0 \neq 0$. If $r_0 = 0$, then $p \in [2\sqrt{q_0}, +\infty)$, i.e. p is unbounded.

27. Let a, b, c be non-negative real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that

$$(a-1)(b-1)(c-1) \geqslant 8.$$

Solution. The desired inequality can be written as

$$r - q + p - 9 \geqslant 0$$

and the relation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ can be written as $q = r$. From relation it follows that $r \neq 0$. Fix q and r. Then p attains the minimal value when $a = b$. Substituting $a = b$ and $c = \frac{a}{a-2}$, we get

$$2a^2 - 12a + 18 = 2(a-3)^2 \leqslant 0.$$

Equality is attained when $a = b = c = 3$.

28. Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{9-ab} + \frac{1}{9-bc} + \frac{1}{9-ca} \leqslant \frac{3}{8}.$$

Solution. The desired inequality can be written as

$$-3r^2 + 19rp - 99q + 243 \geqslant 0$$

and the relation $a + b + c = 3$ can be written as $p = 3$. Fix p and r. Then q attains the maximal value when $a = b$. Substituting $a = b$ and $c = 3 - 2a$, we get

$$6a^4 - 9a^3 - 27a^2 + 57a - 27 = 3(a-1)^2(2a^2 + a - 9) \leqslant 0.$$

Since $0 \leqslant a \leqslant \frac{3}{2}$, it follows that $2a^2 + a - 9 < 0$. Equality is attained when $a = b = c = 1$.

29. Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geqslant \frac{2}{1+abc}.$$

Solution. The desired inequality can be written as

$$4r^2p + 4rq - 4rp - 16r^2 + 3r + 1 \geqslant 0$$

and the relation $a + b + c = 3$ can be written as $p = 3$. Fix p and r. Then q attains the maximal value when $a = b$. Substituting $a = b$ and $c = 3 - 2a$, we get

$$4a^4 - 12a^3 + 13a^2 - 6a + 1 = (2a-1)^2(a-1)^2 \geqslant 0.$$

Equality is attained when $a = b = c = 1$, $a = b = \frac{1}{2}$ and $c = 2$.

30. Let a, b, c be non-negative real numbers such that $a+b+c = 4$ and $a^2+b^2+c^2 = 6$. Prove that

$$a^6 + b^6 + c^6 \leqslant a^5 + b^5 + c^5 + 32.$$

Solution. Using condition, we get $p = 4$ and $q = 5$. Now, by problem 8,

$$a^5 + b^5 + c^5 = p^5 - 5p^3q + 5pq^2 + 5p^2r - 5qr,$$

$$a^6 + b^6 + c^6 = p^6 - 6p^4q + 6p^3r + 9p^2q^2 - 12pqr - 2q^3 + 3r^2.$$

Substituting these equalities in the desired inequality, we get inequality that can be easily proved. Equality is attained when $a = b = 1$ and $c = 2$.

31. Let a, b, c be non-negative real numbers such that $ab + bc + ca \neq 0$. Prove that

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geqslant \frac{10}{(a+b+c)^2}.$$

Solution. The desired inequality can be written as

$$\frac{p^4 - 4p^2q + 5q^2 - 2pr}{q^2p^2 - 2q^3 - 2p^3r + 4pqr - r^2} \geqslant \frac{10}{p^2} \Leftrightarrow$$

$$\Leftrightarrow 10r^2 + r(18p^3 - 40pq) - 10p^2q^2 + 20q^3 + p^6 - 4p^3q + 5pq^2 \geqslant 0.$$

Since $p^2 \geqslant 3q$, it follows that $18p^3 - 40pq \geqslant 0$ and the left hand side attains the minimal value when r is minimal.

- If $a = b$ and $t = \frac{a}{c}$, then the desired inequality can be written as

$$20t^3 - 11t^2 + 4t + 1 \geqslant 0.$$

- If $a = 0$ and $t = \frac{b}{c}$, then the desired inequality can be written as

$$(t-1)^2(t^4 + 4t^3 + t^2 + 4t + 1) \geqslant 0.$$

  Equality is attained when $a = 0$, $b = c$.

32. Let a, b, c be non-negative real numbers. Prove that

$$a^5 + b^5 + c^5 + abc(ab + bc + ca) \geqslant a^2b^2(a+b) + b^2c^2(b+c) + c^2a^2(c+a).$$

Solution. The desired inequality can be written as

$$p^5 - 5p^3q + 7p^2r + 4pq^2 - 3qr \geqslant 0.$$

If $a = b = c = 0$, then inequality turns out to be equality. Fix p and q. Since $7p^2 > 3q$, it follows that left hand side is linear in r and the minimal value is attained when r is minimal.

- If $a = 0$, then the desired inequality can be written as

$$(b^2 - c^2)(b^3 - c^3) = (b-c)^2(b+c)(b^2 + bc + c^2) \geqslant 0.$$

  Equality is attained when $a = 0$, $b = c$.

- If $a = b$, then the desired inequality can be written as

$$c(c^2 - a^2)^2 \geqslant 0.$$

  Equality is attained when $a = b = c$.

33. a) Let a, b, c be non-negative real numbers. Prove that

$$\frac{a^4 + b^4 + c^4}{ab + bc + ca} + \frac{3abc}{a + b + c} \geqslant \frac{2}{3}(a^2 + b^2 + c^2).$$

Solution. The desired inequality can be written as

$$3p^5 - 14p^3q + 8q^2p + 12p^2r + 9qr \geqslant 0.$$

If $a = b = c = 0$, then inequality turns out to be equality. Fix p and q. Left hand side is linear in r and the minimal value is attained when r is minimal.

- If $a = 0$ and $t = \frac{b}{c}$, then the desired inequality can be written as

$$3t^4 - 2t^3 - 2t + 4 \geqslant 0.$$

- If $a = b$ and $t = \frac{a}{c}$, then the desired inequality can be written as

$$4t^5 - 5t^4 + 6t^3 - 10t^2 + 2t + 3 = (t-1)^2(4t^3 + 3t^2 + 8t + 3) \geqslant 0.$$

  Equality is attained when $a = b = c$.

b) Find the least non-negative real k such that the inequality

$$k\frac{a^4 + b^4 + c^4}{ab + bc + ca} + (1-k)\frac{3abc}{a+b+c} \geqslant \frac{a^2 + b^2 + c^2}{3}$$

holds for all non-negative numbers a, b, c.

Solution. The solution is similar to the previous problem.

34. For positive real numbers a, b, c define $X = \frac{a^2+b^2}{2c^2} + \frac{b^2+c^2}{2a^2} + \frac{c^2+a^2}{2b^2}$, $Y = \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}$. Prove that

$$4X + 69 \geqslant 27Y.$$

Solution. The desired inequality can be written as

$$f(p,q,r) = -225r^3 - 50p^3r^2 + 4q^3r + 55pqr^2 + 6p^2q^2r - 4p^4qr - 4pq^4 + 2p^3q^3 \geqslant 0. \quad (*)$$

Fix $p = p_0$ and $q = q_0$. Then $f''(p_0, q_0, r) = -1350r - 100p_0^3 + 110p_0q_0$. Let us prove that

$$-1350r - 100p^3 + 110pq \leqslant 0.$$

Fix p and r. The left hand side is linear in q. So q attains the minimal value when $a = b$. Since inequality is homogeneous, we see that it is enough to prove inequality when $a = b = 1$. Substituting $a = b = 1$ in the desired inequality, we get

$$-20(5c^3 + 19c^2 + 100c + 29) \leqslant 0,$$

which is correct.

From $f''(p_0, q_0, r) \leqslant 0$ it follows that f is concave. So it attains the minimal value when r is either minimal or maximal.

- Substituting $a = 0$ in $(*)$, we get

$$pq^3(2p^2 - 4q) \geqslant 0.$$

- Substituting $a = b = 1$ in the desired inequality, we get

$$(c-2)^2(c-1)^2(4c+1) \geqslant 0.$$

  Equality is attained when $2a = 2b = c$ and $a = b = c$.

35. a) Let $P(a,b,c)$ be a homogeneous symmetric polynomial of degree not greater than 8. Find an algorithm checking whether P is non-negative when a,b,c are non-negative. We assume that we are ale to find extrema and zeroes of an arbitrary function in one variable.

Solution. Without loss of generality, the degree of the polynomial $P(a,b,c)$ is equal to 8. If $a=0$ and $t=\frac{c}{b}$, then

$$P(0,b,c)\geqslant 0 \;\Leftrightarrow\; P(0,0,c)\geqslant 0,\; P(0,1,t)\geqslant 0.$$

If $a=b$ and $t=\frac{c}{a}$, then

$$P(a,a,c)\geqslant 0 \;\Leftrightarrow\; P(0,0,c)\geqslant 0,\; P(1,1,t)\geqslant 0.$$

Consider an inequality

$$P(a,b,c)=A(q)r^2+B(q)r+C(q)\geqslant 0$$

Fix q such that $A(q)\leqslant 0$. Then $P(a,b,c)\geqslant 0 \Leftrightarrow P(a,a,c)\geqslant 0,\ P(0,b,c)\geqslant 0$. Fix q such that $A(q)>0$. The left hand side is minimal when $r=r_0=-\frac{B(q)}{2A(q)}$. Since $T(p,q,r)\geqslant 0$, it follows that r belongs to a segment $I=\left[\frac{9q-2}{27}-\frac{2-6q}{27}\sqrt{1-3q},\frac{9q-2}{27}+\frac{2-6q}{27}\sqrt{1-3q}\right]$. If $r_0\notin I$, then

$$P(a,b,c)\geqslant 0 \Leftrightarrow P(a,a,c)\geqslant 0, P(0,b,c)\geqslant 0.$$

If $r_0\in I$, then

$$P(a,b,c)\geqslant 0 \Leftrightarrow A(q)r_0^2+B(q)r_0+C(q)\geqslant 0 \Leftrightarrow 4A(q)C(q)-B^2(q)\geqslant 0. \quad (*)$$

Finally, we have the following algorithm:

1. Check that $P(0,b,c)\geqslant 0$.
2. Check that $P(a,a,c)\geqslant 0$.
3. Express an inequality $P(a,b,c)\geqslant 0$ in terms of p, q, r and substitute $p=1$.
4. Find q such that $A(q)>0$.
5. Find q such that $r_0=-\frac{B(q)}{2A(q)}\in I$.
6. For q belonging to the intersection of sets from 4 and 5 check $(*)$.

b) Find an analogous algorithm for a homogeneous symmetric polynomial of degree not greater than 17.

Hint. Recall Cardano's formula. Note also that arithmetical roots can be complex.

c*) Find an analogous algorithm for any homogeneous symmetric polynomial.

36. Let a, b, c be non-negative real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. Prove that

$$2(a+b+c-2)^2+(ab+bc+ca)(2+3(a+b+c))\geqslant 35.$$

Hint. Perform the substitution $a=x^2$, $b=y^2$, $c=z^2$.

37. Let a, b, c be real numbers such that $a,b,c\geqslant 1$ and $a+b+c=9$. Prove that

$$\sqrt{ab+bc+ca}\geqslant\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Hint. Perform the substitution $a=(x+1)^2$, $b=(y+1)^2$, $c=(z+1)^2$.

38. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{b+c}\right)^3+\left(\frac{b}{c+a}\right)^3+\left(\frac{c}{a+b}\right)^3+\frac{13abc}{(a+b)(b+c)(c+a)}\geqslant 2.$$

Hint. Perform the substitution $x=\frac{a}{b+c}$, $y=\frac{b}{a+c}$, $z=\frac{c}{a+b}$. Then

$$xy+yz+zx+2xyz=1.$$

39. Let a, b, c be positive real numbers such that $a^2+b^2+c^2+abc=4$. Prove that

$$\frac{a^2}{4-bc}+\frac{b^2}{4-ca}+\frac{c^2}{4-ab}\leqslant 1.$$

Solution. The desired inequality can be written as

$$abc(a^3+b^3+c^3)-4(a^2+b^2+c^2)(ab+bc+ca)+a^2b^2c^2+16(a^2+b^2+c^2)+16(ab+bc+ca)-64\leqslant 0.$$

Since $a^2+b^2+c^2=4-abc$, we get

$$abc(a^3+b^3+c^3+4(ab+bc+ca)+abc-16)\leqslant 0.$$

Consider the case $a=0$. Equality is attained when $a=0$, $b^2+c^2=4$. Otherwise dividing both sides by abc, we obtain

$$a^3+b^3+c^3\leqslant 16-abc-4(ab+ac+bc).\quad(*)$$

Substituting $x-2$ for a, $y-2$ for b, $z-2$ for c in (*) and the relation $a^2+b^2+c^2+abc=4$, where $x,y,z>2$, we get

$$p^3-3pq+4r\leqslant 6p^2-14q$$

and the relation $r=4q-p^2$. Multiplying both sides by $4q-p^2$, we get the homogeneous inequality

$$(p^3-3pq+4r)(4q-p^2)\leqslant(6p^2-14q)r.$$

It is equivalent to the inequality

$$0\leqslant r(10p^2-30q)-(p^3-3pq)(4q-p^2).$$

The coefficient of r equals 0 if and only if $x=y=z$, i.e. $a=b=c$. Therefore equality in the desired inequality is attained when $a=b=c=1$. Otherwise the right hand side is minimal when r is minimal.

- If $x = 0$, then $a = 2$. Now if we recall $a^2 + b^2 + c^2 + abc = 4$, we get $b = c = 0$. This solution has been found before.

- If $x = y$, then
$$z = 4x - x^2 = 4(a+2) - (a+2)^2 = 4 - a^2.$$
Therefore $a = b$, $c = 2 - a^2$. If we combine this with (*), we obtain
$$0 \leqslant (a-1)^2(a+2)^2(a^2 - 2a + 2).$$
Equality is attained when $a = b = c = 1$.

40. Let a, b, c be real numbers. Prove that
$$(a^2 + b^2 + c^2)^2 \geqslant 3(a^3b + b^3c + c^3a).$$
Equality holds if either $a = b = c$ or
$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}, \quad \frac{b}{\sin^2 \frac{4\pi}{7}} = \frac{c}{\sin^2 \frac{2\pi}{7}} = \frac{a}{\sin^2 \frac{\pi}{7}},$$
$$\frac{c}{\sin^2 \frac{4\pi}{7}} = \frac{a}{\sin^2 \frac{2\pi}{7}} = \frac{b}{\sin^2 \frac{\pi}{7}}.$$
Hint. Perform the substitution $a = x + 2\cos\frac{\pi}{7}y$, $b = y + 2\cos\frac{\pi}{7}z$, $c = z + 2\cos\frac{\pi}{7}x$. Then we get homogeneous inequality.

But there exists another solutions. By definition put
$$f(a,b,c) = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a).$$
Then either
$$f(a,b,c) = \frac{1}{2}(a^2 - 2ab + bc - c^2 + ac)^2 + \frac{1}{2}(b^2 - 2bc + ac - a^2 + ab)^2 +$$
$$+\frac{1}{2}(c^2 - 2ac + ab - b^2 + bc)^2,$$
or
$$f(a,b,c) = \frac{1}{4}(a^2 + b^2 - 3ab + 3ac - 2c^2)^2 + \frac{3}{4}(a^2 - ab - ac - b^2 + 2bc)^2.$$

41. Find conditions on numbers p, q, r necessary and sufficient for

a) numbers a, b, c to be not less than 1;

Solution. It is an obvious corollary of the problem 23. $T(p,q,r) \geqslant 0$ and
$$(a-1) + (b-1) + (c-1) \geqslant 0 \Leftrightarrow p \geqslant 3,$$
$$(a-1)(b-1) + (b-1)(c-1) + (c-1)(a-1) \geqslant 0 \Leftrightarrow q - 2p + 3 \geqslant 0,$$

$$(a-1)(b-1)(c-1) \geqslant 0 \Leftrightarrow r-q+p-1 \geqslant 0.$$

b) numbers a, b, c to be side lengths of a triangle (perhaps degenerate);

Hint. Use the problem 23 for numbers $(a+b-c)$, $(a-b+c)$, $(-a+b+c)$.

c) non-negative real numbers a, b, c to satisfy $2\min(a,b,c) \geqslant \max(a,b,c)$.

Hint. $2\min(a,b,c) \geqslant \max(a,b,c) \Leftrightarrow 2a \geqslant b,\ 2a \geqslant c,\ 2b \geqslant a,\ 2b \geqslant c,\ 2c \geqslant a,\ 2c \geqslant b \Leftrightarrow$ $\Leftrightarrow a \geqslant 0,\ b \geqslant 0,\ c \geqslant 0,\ (2a-b)(2b-a) \geqslant 0,\ (2b-c)(2c-b) \geqslant 0,\ (2a-c)(2c-a) \geqslant 0.$

42. Let a, b, c be real numbers such that $a,\ b,\ c \in \left[\frac{1}{3}, 3\right]$. Prove that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geqslant \frac{7}{5}.$$

Hint. Perform the substitution $x = \frac{b}{a}$, $y = \frac{c}{b}$, $z = \frac{a}{c}$.

43. Let a, b, c be side lengths of a triangle. Prove that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geqslant 6\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a}\right).$$

44. Prove that there exists a polynomial $S(x,y,z)$ such that the following conditions are equivalent: **(i)** a, b, c are real numbers; **(ii)** $S(x,y,z) \geqslant 0$, where $x = a+b+c$, $y = a^2+b^2+c^2$, $z = a^3+b^3+c^3$.

Solution.

$$S(x,y,z) = T(p,q,r) = T\left(x, \frac{x^2-y}{2}, \frac{x^3-3xy+2z}{2}\right).$$

45. Prove that the following conditions are equivalent: **(i)** $s \in \left[\frac{86}{9}, 10\right]$; **(ii)** there exist real numbers a, b, c such that $a+b+c = 4, a^2+b^2+c^2 = 6, a^3+b^3+c^3 = s$.

Solution. It is an obvious corollary of the previous problem.

46. (USA TST, 2001) Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2+abc = 4$. Prove that

$$ab+bc+ca-abc \leqslant 2.$$

Hint. $x = a+b+c = p$, $y = a^2+b^2+c^2 = p^2-2q$, $z = abc$.

47. (China-West, 2004) Let a, b, c be positive real numbers. Prove that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \leqslant \frac{3}{2}\sqrt{2}.$$

Hint. Perform the substitution $x = \sqrt{1+\frac{b^2}{a^2}}$, $y = \sqrt{1+\frac{c^2}{b^2}}$, $z = \sqrt{1+\frac{a^2}{c^2}}$.

48. Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2+nabc = n+3$ for some real n.

a) Assume that $0 \leqslant n \leqslant \frac{3}{2}$. Prove that $a+b+c \leqslant 3$.

b) Assume that $\frac{3}{2} \leqslant n \leqslant 2$. Prove that $a+b+c \leqslant \sqrt{2(n+3)}$.

Hint. $x = a+b+c$, $y = a^2+b^2+c^2+nabc$, $z = abc$.

c) Assume that $n = 2$. Prove that $ab+bc+ac-abc \leqslant \frac{5}{2}$.

Hint. The proof is simular to the proof of the problem 46.

49. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2+abc = 4$. Prove that

$$a^2b^2+b^2c^2+c^2a^2 \geqslant abc(2+\sqrt{4-3abc}).$$

Hint. $x = a^2+b^2+c^2+abc = 4$, $y = ab+bc+ca$, $z = abc$.