Dynamics of Tilings

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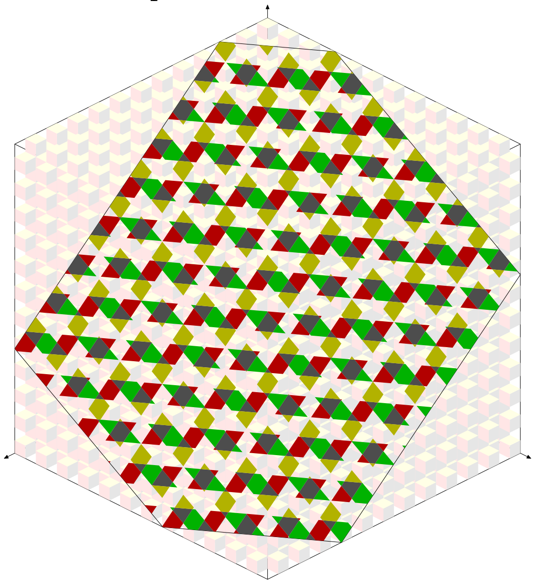
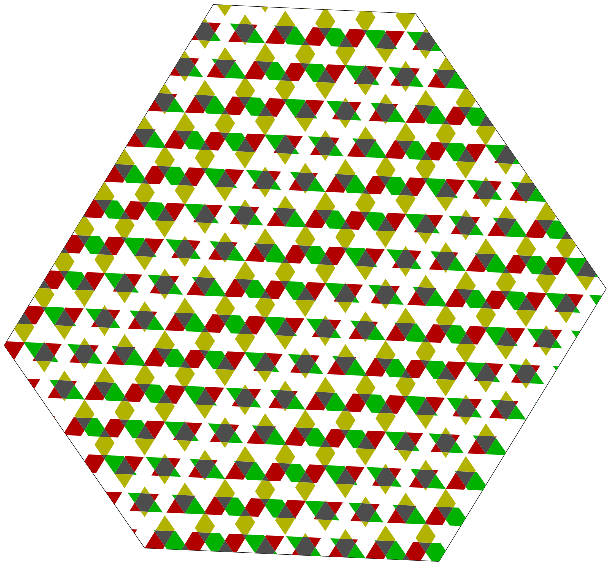
This story is a stroll to an open problem connected with topology, dynamics and even the physics of crystals. The problem is about sets like these, which are called *quasiperiodic*:





To understand the formulation of the problem you must get acquainted with the basic concepts of *ergodic theory* (§2 и §3) and *topology* (§5 и §7– these two paragraphs are independent of the rest, you can start with them).

The central paragraph 4 about tilings on the plane – is an introduction to quasiperiodic geometry, which is the topic of paragraphs §6 and §8. The open problem is at the very end.

Before looking at tilings on the plane, we start out with *tilings on the line*. If you draw a line on checkered paper then the cells will divide the line into segments.

What are the possible lengths of these segments? How often do they occur?

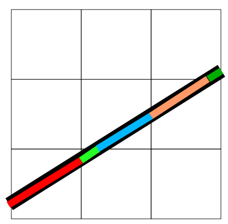
The problems in paragraphs §§1–3 are dedicated to answering these sort of questions.

If you have trouble with some notation or a question, or you just can't seem to solve a problem without a star – ask us (Arina Arkhipova, Polina Belopashenseva, Ilya Bogdanov, Sergey Dorichenko, Alexander Esterov, Kirill Stupakov)!!

§1 Tilings on a rational line

1. **a)** For how many c the line given by the equation $6x + 8y = c$ intersects the square $[0, 1]^2$?
b) What are the lengths of these intersections? How many different lengths are there? How many intersections have a given length?

The line $ax + by = c$ is divided by the cells into segments which we will call *tiles*, a division will be called a *tiling* on a line.



2. **a)** How many different tile lengths occur on line $ax + by = c$ for given integers a, b and c.
b) What is the *period* of this tiling, i.e. the minimal shift required to move the tiling along the line so that it translates to itself.
c) From a corner of a pool table with integer lengths of sides $A \times B$ a ball is hit along the bisector. How many times will it hit the sides of the table before it comes to a corner? What is the minimal and maximal distance the ball travels between two bounces?

§2 Tilings on an irrational line

Here we look at tilings on an *irrational* line $ax + by = c$ such that a/b is an irrational number (e.g. we can study the case $x + \sqrt{2}y = c$).

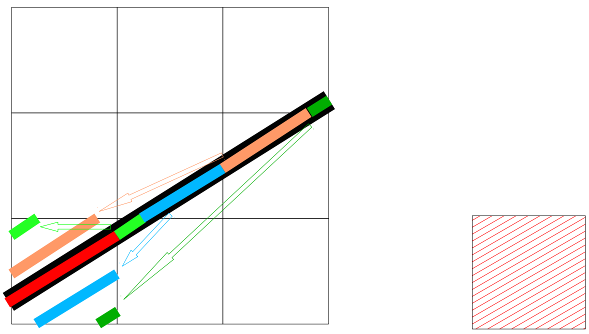
3. **a)** Can there be two tiles of equal length on an irrational line?
b) Can there be or not be an infinite number of tiles with equal length?
c) Can there be or not be exactly three tiles of equal length?
d) Can there be or not be an infinite number of tiles with pairwise different lengths?
e) Describe geometrically all the lines which have a pair of tiles of equal length different from the rest of the tiles.

If you have trouble with **b-e** (and you will!) then first solve the following problem:

4. **a)** Prove that any bounded (contained in a segment) sequence of numbers has arbitrarily close members.

b) **(Dirichlet Lemma)** If λ – is an irrational number, then the sequence $\{\lambda\}$, $\{2\lambda\}$, $\{3\lambda\}$, $\{4\lambda\}, \ldots$ (where $\{\}$ denotes the decimal part) is dense in the segment $[0, 1]$, i.e. every subsegment contains at least one member of the sequence.

c) Cut up the checkered paper, with the line $ax+by = c$ drawn on it, into cells, put all of them in a stack, and look at it towards a light. The visible subset of the square $[0, 1]^2$ is called the winding of the square $[0, 1]^2$ with the line $ax+by = c$ (see the picture below). Define the winding rigorously and prove that it intersects with every segment, which is not parallel to the line $ax+by = c$. Now you can solve the previous problem.



§3 Limits and averages

Recall that a number a is called the limit of a sequence a_1, a_2, $a_3, \ldots$, if each neighborhood of a (i.e. each open interval containing a) contains all but a finite number of the members of the sequence. In this case we write $a = \lim_{n\to\infty} a_n$ and say that the sequence a_n *converges* to a. If you know this **very well** then you can skip the next problem.

5. Prove that **a)** $\lim_{n\to\infty} 1/n = 0$, **b)** the sequence $a_n = (-1)^n$ doesn't converge,
c) if $c_n = a_n + b_n$, and a_n and b_n converge to a and b respectively the c_n converges to $a + b$.

d) (Squeeze lemma or Two Policeman and a Drunk lemma or Sandwich lemma) If $a_n < c_n < b_n$ and a_n and b_n converge to a, then c_n converges to a.

6. From the corner of a pool table with an irrational ratio of sides $A \times B$ a ball is hit along the bisector with uniform velocity. **a)** Find the average number of bounces i.e.

$$\lim_{T\to\infty} \frac{\text{number of bounce in the first } T \text{ minutes}}{T}.$$

b) Find the frequency of doublets i.e. pairs of sequential bounces from parallel sides.

If you have trouble with **b** (you might not!) return to it at the end of the paragraph.
We will call *frequency of hits* of a sequence a_n in a segment I the limit

$$\lim_{N\to\infty} \frac{\text{number of integer } n < N, \text{ such that } a_n \text{ lies in } I}{N}.$$

Your task in the next problem is to prove **Weyl's Theorem**:

> For an irrational number λ and any a the frequency of hits of the sequence $\{a + n\lambda\}$ in a subsegment I of $[0, 1]$ is equal to the length of I.

We will call this sequence a *Weyl progression with difference* λ.

7. **a)** Given a number ε and a segment I, find some irrational λ and integer N, such that

$$\frac{\text{number of integer } n < N, \text{ such that } \{a + n\lambda\} \text{ lies in } I}{N}$$

differs from the length of I by no more than ε.

Tip: take a very small λ and a very big N.

b) Prove that all N starting from some N_0 suit the needs of **a**.

c) Prove that for arbitrarily small δ any Weyl progression can be divided into several subsequences such that each one of them is also a Weyl progression with difference less than δ.

Tip: Apply the Dirichlet lemma to this sequence for the segment $[0, \delta]$.

d) Prove Weyl's Theorem.

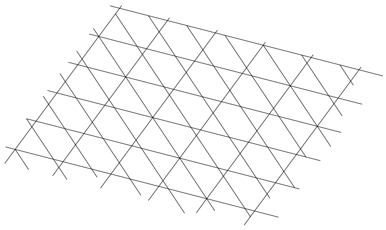
8. **a)** I is a segment inside the square $[0, 1]^2$. Give a definition of the frequency of intersections of the winding of $[0, 1]^2$ with the segment I (the winding of the square was defined at the end of the previous paragraph). **b)** Find this frequency for the case, when I is a horizontal of vertical segment of length d. **c)** Find the frequency for any segment I.

9. **a)** Find the *density* of tiles on the line $ax + by = 0$, i.e.

$$\lim_{T\to\infty} \frac{\text{number of tiles on the segment of length } T, \text{ drawn from the point}(0,0)}{T}.$$

b) Find the *average length* of tiles on the line $ax + by = 0$, i.e.

$$\lim_{T\to\infty} \frac{\text{the sum of the lengths of tiles on a segment of length } T}{\text{number of tiles}}.$$



10. **a)** Give a definition of a tiling and its tiles on a plane $Ax+By+Cz=0$ in space.
b) What is the maximal number of angles a tile can have?
c) Find the number of different tiles on a plane if A, B and C are integers.
d) Show that for the plane $Ax+By+Cz=0$ exactly one of the following holds:
– 0 is the only point on this plane with integer coordinates (in this case the plane is called *irrational*);
– all the points on this plane with integer coordinates are of the form $k \cdot v$ where v – is one of these points and k is an integer number (in this case the plane is called *semirational*);
– $\lambda A, \lambda B$ and λC – are integers for some $\lambda \neq 0$ (*rational* plane).
e) Formulate and solve analogs of the problems from paragraph §2 for the irrational plane.
f) Formulate and solve analogs of the problems from paragraph §2 for the semirational plane.

Halfway finish

Choose some polygon P (convenient for you) in the plane $Ax+By+Cz=0$, containing the point $(0,0,0)$. Denote by P_T its homothety with center at $(0,0,0)$ and coefficient T. This figure will play the same role for the plane, as the segment of length T did on the line in paragraph §3.

11. **a)** Formulate and prove an analog of Weyl's Theorem for the *two parameter* sequence $\{k\mu+n\lambda\}$.
b) Formulate and prove an analog of problem **8** for the irrational plane in space.
c) Define and find the average area of tiles of a given irrational plane.
d) Define and find the density of triangular tiles on a irrational plane.
e) Find the probability that a randomly chosen tile is a triangle, i.e.

$$\lim_{T\to\infty} \frac{\text{number of triangle tiles in } P_T}{\text{number of tiles in } P_T}.$$

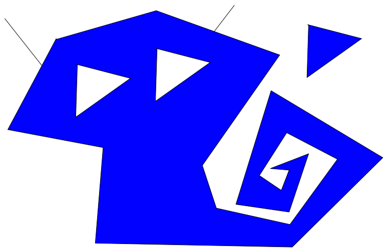
f)$^{\times}$ Find the probability that a randomly chosen point of the plane is in a triangular tile, i.e.

$$\lim_{T\to\infty} \frac{\text{the sum of the areas of the triangular tiles in } P_T}{\text{sum of the area of all tiles in } P_T}.$$

g)$^{\times}$ Define and find the average area of triangular tiles of a given irrational plane.
h) Solve all these problems replacing triangles with four, five, six, seven sided polygons.

§5 Polyhedrons

Recall that a *closed (open) convex polygon* – is an intersection of several half planes including the boundary (respectively not including the boundary). *(Closed) polyhedron* – is a union of several (closed) convex polygons. A *component* of a polyhedron P is a set of all its points that can be connected by piecewise linear curve inside P. Here is an example of a polyhedron with two components, while the complement to it has three components:



12. a) Prove that the union and interesction of (closed) polyhedrons, and the connected component of a (closed) polyhedron – is also a (closed) polyhedron.
b) Prove that the complement to a closed polyhedron – is also a polyhedron, but not closed.
c) If we know the number of components of polyhedrons P and Q, what can we say about the number of components of their union? Intersection? Complement to P?

A subset of the plane is called *periodic* if it does not change under integer translations in the horizontal and vertical directions. The *fundamental domain* of a periodic set is its intersection with the unit square. We will call a set a *periodic polyhedron* if it is periodic and its fundamental domain is a polyhedron.

13. a) Can different periodic sets have the same fundamental domain?
b) Can a periodic polyhedron have one connected component, and its fundamental domain two ?
c)$^{\times}$ Can a periodic polyhedron have two connected components?

14. If two closed non self intersecting piecewise linear curves don't pass through each other's vertices, then they intersect in an even number of points. Prove this **a)** for two triangles **b)** when one of the curves is a triangle **c)** in the general case.

Warning: if you plan to use concepts such as *"inside "* and *"outside a closed non self intersecting piecewise linaer curve"* then you will have to give a definition and prove the properties you plan to use. (But this problem can be solved even easier without using these concepts).

15. Two vertices of a connected graph A on the plane got connected with a new edge, which does not pass through any of the other edges or vertices. Prove that the number of connected components of the complement to this graph increased **a)** by no more than 1 **b)** by no less than 1.

§6 Quasiperiodic sets

A *quasiperiodic set* on the line $ax + by = c$ is the intersection of this line with a periodic polyhedron.

16. a) Can a finite set on an irrational line be quasiperiodic?
b) If a quasiperiodic set is contained in a ray, must it be finite?
c) Find a subset of an irrational line which is not quasiperiodic.

The measure of a set on line – is its one dimensional area. Formally, if a set on line is a disjoint finite union of segments, then its *measure* is the sum of lengths of all the segments. Define the *average length* of a subset Q of a line as the limit

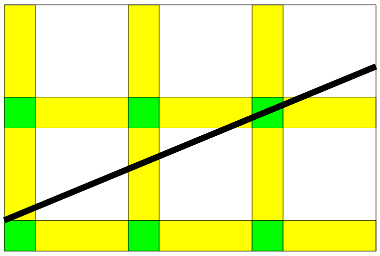
$$\lim_{T\to\infty} \frac{\text{measure of the intersection of } Q \text{ with an segment of length } T, \text{drawn from the point} (0,0)}{T}.$$

17. Let the quasiperiodic set Q on the line $ax+by=c$ be the intersection of this line with a periodic polyhedron P which has fundamental domain F.
a) Find the average length of the set Q, if F is a rectangle, such that one of its sides is parallel to the line $ax+by=c$.
b) Find the average length of the set Q for any F.

Tip: approximate F with rectangles from inside and outside.

Remark: You can now try again to solve the stars in **11**.

For clarity, from now on we will study the quasiperiodic set M_h of the points having at least one coordinate with the fractional part not exceeding h. It is green and yellow in the picture:



Denote by Q_h the intersection of M_h with the line $x+\sqrt{2}y=0$

18. a) Find the average length of Q_h. **b)** Define and find the density of segment of the set Q_h.

Tip: First solve **(b)** for the sets $M_{1,h}$ and $M_{2,h}$, which are green and yellow respectively on the picture. Then express the answer for M_h using the answers for $M_{1,h}$ and $M_{2,h}$.

§7 Euler Characteristic

Using the tip to the previous problem, we make an obvious remark: if we divide a subset of the line into open intervals and points, then the difference between the number of points and number of intervals does not depend on the division. This difference is called the *Euler characteristic* of a subset of the line.

To start studying quasiperiodic sets on the plane, we must first generalize the concept of Euler characteristic to subsets of the plane.

We will call a *cell division* of a polyhedron P a set of open convex polygons T_1, T_2, $T_3, \ldots$ and open intervals I_1, I_2, $I_3, \ldots$ and points P_1, P_2, $P_3, \ldots$ (all together called the *cells* of the division) such that
– each point of the polyhedron belongs to exactly one cell, i.e. each point either lies in one of the polygons (but not in its boundary), or in one of intervals (but not at its endpoint), or is one of the points.
– each side and vertex of each polygon and each end of each interval are also cells.
– the number of cells is finite, and the union of the cells equals the polyhedron P.

19. a) Consider the closed square $[-1,1]^2$ from which the open square $(0,1)^2$ was removed. Show that the resulting set is a closed polyhedron and find its cell division with the least possible number of cells. Is it possible that the triangle with vertices $(-1,-1)$, $(1,-1)$ and $(-1,1)$ is a cell in some cell division of this polyhedron?
b) Prove that each closed polyhedron admits a cell division.

The *Euler characteristic* of a cell division of a polyhedron P is
$\chi(P) = $(number of polygons)$-$(number of intervals)$+$(number of points).

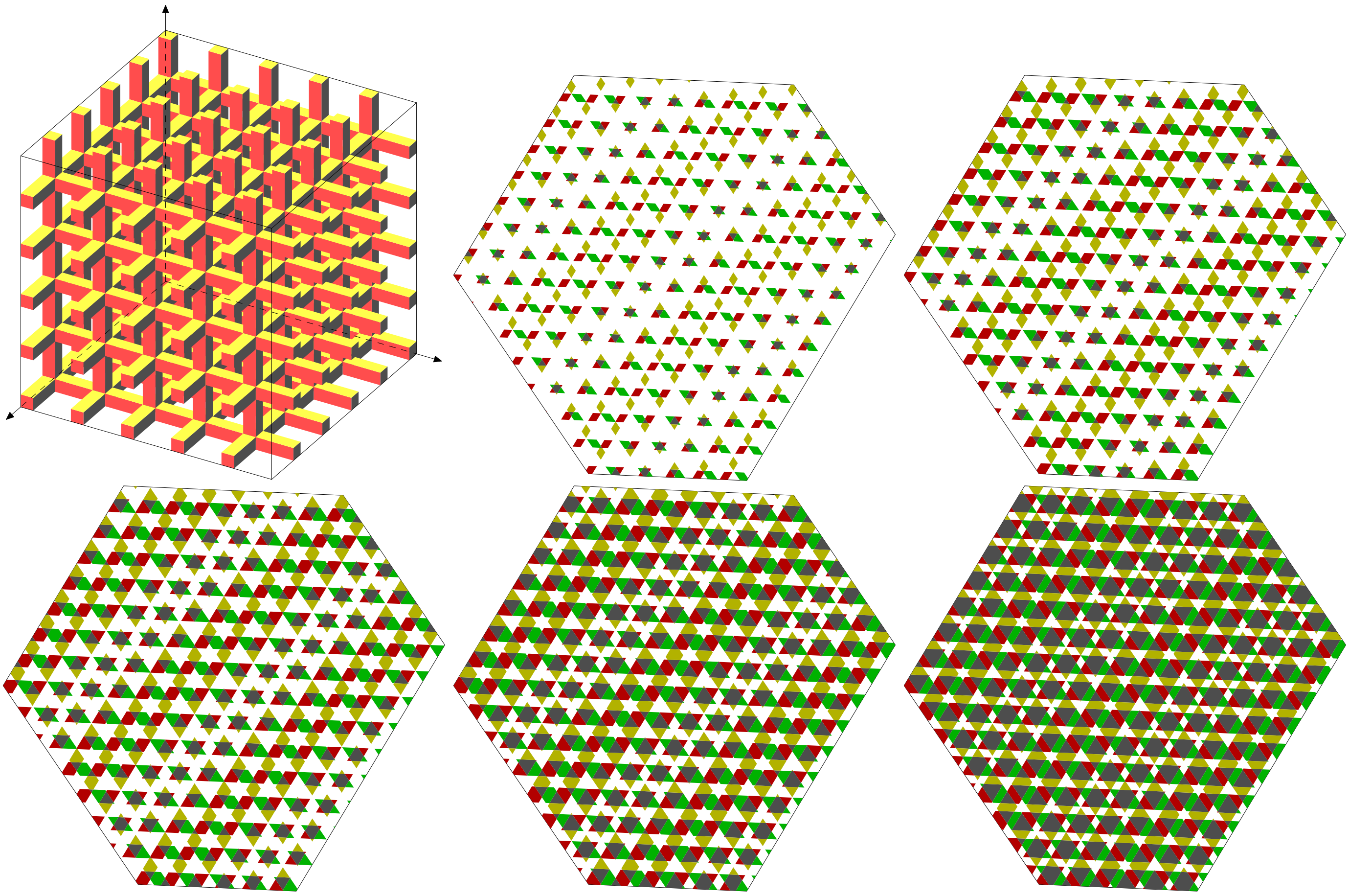
20. a) (Euler Formula) Euler characteristic of a bounded closed polyhedron is equal to: (number of its components)$-$(number of components of its complement)$+1$.

Tip: Prove this for trees, then for graphs, then for a arbitrary polyhedron by adding cells one by one.

b) Euler characteristic of a closed polyhedron does not depend on the choice of the cell division. The Euler characteristic of a convex polygon is 1. **c)** $\chi(P)+\chi(Q)=\chi(P\cup Q)+\chi(P\cap Q)$.
d) (Jordan's Lemma) The complement to a closed non self intersecting piecewise linear curve has exactly two components.

§8 Quasiperiodic sets on the plane

A *quasiperiodic set* on the plane $Ax+By+Cz=0$ is the intersection of this plane with a periodic set in space. In particular, denote by Q_h the intersection of the plane $x+\sqrt[3]{2}y+\sqrt[4]{2}z=0$ with the set M_h, made of the points such that the decimal parts of at least two of the three coordinates do not exceed h. For instance, the set $M_{1/4}$ and the sets Q_h for $h=0.3$, 0.4, 0.5, 0.6, 0.7 look as follows:



As in problem **11**, choose some (convenient for you) polygon P in $x+\sqrt[3]{2}y+\sqrt[4]{2}z=0$, denote by P_T its homothetic image with center (0,0,0) and coefficient T.

Open problem. Calculate the *component density* of the quasiperiodic set Q, i.e. limit

$$\sharp Q = \lim_{T\to\infty} \frac{\text{number of components } Q \text{ в } P_T}{\text{area } P_T}.$$

The answer is not know even for sets Q_h shown above for arbitrory h.

21. a) Try to find the least possible $h_0 < 1$, for which $\sharp Q_{h_0} = 0$. (The series of pictures above will help you guess, perhaps not the smallest, but some acceptable h_0)

b) Calculate the density of the Euler characteristic of the set Q_h, i.e.

$$\lim_{T\to\infty} \frac{\text{Euler characteristic of the intersection } Q \text{ and } P_T}{\text{area of } P_T}.$$

c) Show that the complement to the set M_h can be produced from M_{1-h} by removing the boundary points and applying a central symmetry with respect to 0.

d) How are $\sharp Q_h$ and $\sharp Q_{1-h}$ related?

e) Find $\sharp Q_h$ for sufficiently large and small h, specifically when $h \geqslant h_0$ and $h \leqslant 1 - h_0$.

f) Give upper and lower estimations for $\sharp Q_{1/2}$. Try to make them as close as you can.

g) Propose H as large as possible, for which $\sharp Q_H > 0$, and give a lower estimation for this density.

h) Consider the intersection of M_h with another plane: $x + \sqrt{2}y + \sqrt{3}z = 0$. This quasiperiodic set $\tilde{Q}_h$ is depicted below for $h = 0.4, 0.5, 0.6$. Find any estimates for $\sharp\tilde{Q}_h$, which would support or cast doubt on the following conjecture: unlike $\sharp Q_{1/2}$, the density of $\sharp\tilde{Q}_{1/2}$ is zero, but $\sharp\tilde{Q}_h > 0$ for $h < 1/2$.

i) (Discrete version of Arnold's problem 1988-17) Find any estimates which would support or cast doubt on the following conjecture, for the largest possible $h < 1/2$: for each h there exists a sufficiently large number R_h, such that each bounded component of the set $\tilde{Q}_h$ belongs to a disk of radius R_h.

The most accurate estimation of $\sharp Q_H$ or $\sharp\tilde{Q}_{1/2}$ will get a prize, an exact calculation of these densities or solution of the discrete Arnold's problem 1988-17 can be published in a mathematical journal, as soon as you acquire the necessary culture for writing mathematical articles, while studying at a math faculty!