The Nagel, Gergonne, and Feuerbach points and their properties

2. Main problems

A.Yakubov, A.Zaykov, M.Didin, P.Kozhevnikov, D.Krekov, A.Zaslavsky, O..Zaslavsky

14. A_0B_0, C_0I, CC'' are concurrent.

Suppose P be an arbitrary point not lying on the sidelines of ABC. Then lines symmetric to AP, BP, CP in bisectors of angles A, B, C, respectively, have a common point (perhaps a point at infinity). This point is called *isogonally conjugate* to P with respect to ABC.

Further notation. Let I_1, I_{A_1}, I_{B_1}, I_{C_1}, H_1 be isotomically conjugates to I, I_A, I_B, I_C, H, respectively; let G_2, N_2 be isogonally conjugates to G, N, respectively. Similarly define G_{A_2}, N_{A_2}, G_{B_2}, N_{B_2}, G_{C_2}, N_{C_2}. Let L' be the Lemoinne point of $\triangle A'B'C'$.

15. N_2 and G_2 are centers of homotheties taking the incircle of ABC to the circumcircle of ABC.

16. H_1, I_1, N, G are collinear and form a harmonic quadruple.

16'. Same for quadruples H_1, I_{A_1}, N_A, G_A, etc.

17. $I_{A_1}I_1$, NN_A, BC are concurrent.

17'. $I_{B_1}I_{C_1}$, N_BN_C, BC are concurrent.

18. $I_{A_1}I_1$, NN_A, BC, GG_A are concurrent.

18'. $I_{B_1}I_{C_1}$, N_BN_C, BC, G_BG_C are concurrent.

19. NG, N_AG_A, N_BG_B, N_CG_C meet at L'.
Corollary: Triangles $N_AN_BN_C$ and $G_AG_BG_C$ are perspective with perspector L'.

20. Lines IL and NG are parallel.

20'. $N_AG_A \parallel I_AL$, $N_BG_B \parallel I_BL$, $N_CG_C \parallel I_CL$.

Let XYZ and $X_1Y_1Z_1$ be perspective triangles. By Desargue theorem, $XY \cap$

X_1Y_1, $XZ \cap X_1Z_1$, $ZY \cap Z_1Y_1$ lie on a line called *the perspective axis* of given triangles.

21. The perspective axis of $N_AN_BN_C$ and ABC coincides with the perspective axis of $G_AG_BG_C$ and ABC. This axis if perpendicular to IG.

(*Sondat theorem*). Suppose triangles XYZ and $X_1Y_1Z_1$ are perspective and orthologic simultaneously. Then two centers of orthology and the perspector lie on a line одновременно и перспективны и ортогологичны, то два центра ортологичности и центр перспективы этих треугольников лежат на одной прямой, перпендикулярной оси перспективы $\triangle XYZ$ и $\triangle X_1Y_1Z_1$

22. I is the orthology center of triangles $N_AN_BN_C$ and ABC.

23. Solve the problem 20 using problems 21 and 22 (and perhaps some of previous).

3. Additional problems

Let U be an arbitrary point not lying on XY, YZ, ZX. The perspective axis of traingle XYZ and the cevian traingle of U is called *the trilinear polar line* of U with respect to $\triangle XYZ$.

24. The trilinear polar line of G is perpendicular to IG.

25. Let U be a point of the circumcircle of $\triangle XYZ$, $U \neq X$,$U \neq Y$,$U \neq Z$. Suppose L_0 is the Lemoinne point of triangle $\triangle XYZ$. Then the trilinear polar line of U with respect to $\triangle XYZ$ passes through L_0.

26. Given a triangle XYZ and a point Q not lying on the sidelines of XYZ. Suppose that $XQ \cap YZ = X_1$;$YQ \cap XZ = Y_1$; $ZQ \cap YX = Z_1$; $Y_1Z_1 \cap YZ = X_2$; then Y,Z,X_1,X_2 is a harmonic quadruple.

27. Let U and V be points not lying on XY, XZ, YZ. Let U' and V' be its isogonally (or isotomically) conjugates with respect to $\triangle XYZ$. If V lies on the trilinear polar line of U', then U lies on the trilinear polar line of V'.

28. Perspective axis of $N_AN_BN_C$ and ABC, or $G_AG_BG_C$ and ABC, is the trilinear polar line of I_1 with respect to ABC.

29. The trilinear polar lines of I_1 and G with respect to ABC are parallel.

30. Solve the problem 20 using problems 24-28 (and perhaps some other previous problems) WITHOUT applying Sondat theorem.

31. If P lies on the trilinear polar line of G, then the trilinear polar line of P touches the incircle.

The Feuerbach theorem. The nine-point circle of triangle ABC touches its incircle and excircles. The touching points F, F_A, F_B, F_C are called *the Feuerbach points.*

31'. Suppose P is the point at infinity of the trilinear polar line of G; then the trilinear polar line of P touches the incircle at F.

32. The reflections of F in the sidelines of $\triangle A_0B_0C_0$ lies on OI.

33. A_A, B_B, C_C, and F are concyclic.

34. Formulate analogues of problems 31', 32, 33 for points F_A, F_B, F_C.