A.Zaslavsky, A.Akopyan,
P.Kozhevnikov, A.Zaslavsky

Brocard points
Solutions

1 Brocard points in triangles

1. Since $\angle PAB = \angle PBC$ we have $\angle BPA = \pi - \angle B$, i.e. P lies on the circle passing through A and B and touching BC. Since $\angle PBC = \angle PCA$, P lies on the circle passing through B and C and touching CA. Therefore P is the common point of these circles distinct from B. Point Q is constructed similarly.

2. Answer $\mathrm{ctg}\phi = \mathrm{ctg}A + \mathrm{ctg}B + \mathrm{ctg}C$ follows rom the formula which will be proved later.

3. Let A', B', C' be the reflections of P in BC, CA, AB. Since $CA' = CP = CB'$ and $\angle PCA = \angle QCB$, we obtain that CQ is the perpendicular bisector to segment $A'B'$, i.e. Q is the circumcenter of $A'B'C'$. Thus the midpoint of PQ is the center of the circle passing through the projections of P to the sidelines of ABC. Similarly the midpoint of PQ is the center of the circle passing through the projections of Q. It is clear that the radii of these circles are equal.

4. Let AP, BP, CP meet for the second time the circumcircle of ABC in points A', B', C'. Then the arcs BA', CB' and AC' are equal, i.e. triangle $B'C'A'$ is the rotation of ABC around O to angle 2ϕ. Then P is the second Brocard point of $A'B'C'$ and this yields both assertions of the problem.

5. Let C' be a common point of lines AP and BQ. Since the angle between these lines is equal to 2ϕ we obtain by previous problem that C' lie on the circle OPQ. Also it is evident that $OC' \perp AB$. Therefore it is sufficient to prove that $C'L \parallel AB$. Let points A', B' be defined similarly as C'. Since triangles ABC', BCA', and CAB' are similar the distances from A', B', C' to the correspondent sides of ABC are proportional to the lengths of these sides. The ratio from the point of circle OPQ opposite to O to these sides are the same. Thus this point coincide with L.

6. a) , b) **Hint.** Consider the spiral similarities with center P (Q), transforming Q (P) to O.

c) **Answer.** The midpoint of OL, $\pi - 2\phi$.

7. Since $\angle PAC' = \angle QBC' = \phi$ and $\angle PC'A = \angle QC'B$, triangles APC' and BQC' are similar, i.e. $AC'/BC' = AP/BQ$. But from triangles ACP, BCQ we have $AP/\sin\phi = AC\sin A$, $BQ/\sin\phi = BC/\sin B$. Therefore, $AC'/BC' = AC^2/BC^2$ and CC' is a symmedian.

8. This is a partial case of problem 27.

2 Brocard point in quadrilaterals

9. The proof is the same as in problem 1.

10. Since $\angle APB = \pi - \angle B$, $\angle BPC = \pi - \angle C$, we obtain using the sinus theorem to triangles APB and BPC

$$\frac{PB}{\sin\phi} = \frac{AB}{\sin B}, \quad \frac{PB}{\sin(C-\phi)} = \frac{BC}{\sin C}.$$

Dividing the first equation to the second one we have

$$\mathrm{ctg}\phi = \frac{AB}{BC\sin B} + \mathrm{ctg}C.$$

11. The condition $\phi(ABCD) = \phi(DCBA)$ is equivalent to

$$\frac{AB}{BC\sin B} - \mathrm{ctg}B = \frac{CD}{BC\sin C} - \mathrm{ctg}C.$$

Adding the unit to the squares of both parts we obtain

$$\frac{AB^2 + BC^2 - 2AB\cdot BC\cos B}{\sin^2 B} = \frac{CD^2 + BC^2 - 2CD\cdot BC\cos C}{\sin^2 C},$$

i.e. $\frac{AC}{\sin B} = \frac{BD}{\sin C}$, ч.т.д.

12. By the construction of P_i we obtain that quadrilaterals BCP_1P_2, CDP_2P_3, DAP_3P_4, ABP_4P_1 are cyclic. From this $\angle P_1P_4P_3 + \angle P_3P_2P_1 = \angle A + \angle C = \pi$.

13. Since BCP_1P_2 is a cyclic quadrilateral we obtain

$$\frac{P_1P_2}{BC} = \frac{\sin(\phi(ABCD) - \phi(BCDA)}{\sin(C + \phi(ABCD) - \phi(BCDA)} = \frac{Q_1Q_2}{CD}.$$

14. See http://jcgeometry.org/Articles/Volume2/Belev_Brocard_points.pdf

15. See http://jcgeometry.org/Articles/Volume2/Belev_Brocard_points.pdf

16. See http://jcgeometry.org/Articles/Volume2/Belev_Brocard_points.pdf

17. Since $\phi(ABCD) = \phi(DCBA)$, the sought equality is equivalent to

$$\frac{AB}{BC\sin B} + \mathrm{ctg}C = \frac{AD}{DC\sin D} + \mathrm{ctg}C.$$

Since $\sin B = \sin D$ we obtain the assertion of the problem.

18. Since $ABCD$ is cyclic, $AB\cdot CD + AD\cdot BC = AC\cdot BD$, i.e. $AB\cdot CD = AC\cdot BD/2$. Let M be the midpoint of AC. Then $CM\cdot BD = BC\cdot AD$, i.e. $BC/CM = BD/AD$. Since $\angle BCM = \angle BDA$, triangles BCM and BDA are similar. Therefore, $\angle MBC = \angle ABD$ and BD is the symmedian of ABC, which yields a)-c).

For prove d) note that four arbitrary points can be transformed by an inversion to the vertices of a parallelogram. If the given points are concyclic this parallelogram will be a rectangle with the same ratio of the products of the opposite sides. Thus the vertices of a harmonic quadrilateral will be transformed to the vertices of a square.

For prove e) consider a central projection conserving the circumcircle of $ABCD$ and transforming the common point of its diagonals to the center. Then the image of the quadrilateral will be a rectangle. Since the tangents to the circumcircle in the opposite vertices of this rectangle are parallel to its diagonal the rectangle is a square.

19. Since $\mathrm{ctg}\phi = \frac{AB}{BC \sin B} + \mathrm{ctg}C = \frac{BC}{AB \sin B} + \mathrm{ctg}A$, $\mathrm{ctg}^2\phi - \mathrm{ctg}^2 A = \frac{1}{\sin^2 B}$ or

$$\frac{1}{\sin^2 \phi} = \frac{1}{\sin^2 A} + \frac{1}{\sin^2 B}.$$

20. The proof is the same as in problem 4.
21. the proof is the same as in problem 5.

3 Brocard points in polygons

22. Hint. Prove that all lines X_iX_{i+1} are the tangents to the same ellipse.
23. Answer.

$$\frac{OL^2}{R^2} + \mathrm{tg}^2\phi\,\mathrm{tg}^2\frac{\pi}{n} = 1.$$

24. Hint. Consider the rotations around O to $\pm\phi$.
25. The proof is the same as for $n = 4$.
26. The proof is the same as for $n = 3$.
27. Hint. T_1, T_2 are the limit points of the pencil containing the circumcircle of the polygon and the circumcircle of OPQ.