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Brocard points

1 Brocard points in triangles

1. Let a triangle ABC be given. Prove that there exists a unique point P, such that $\angle PAB = \angle PBC = \angle PCA = \phi_1$, and a unique point Q, such that $\angle QBA = \angle QCB = \angle QAC = \phi_2$.

Definition 1. Points P and Q are called the *Brocard points* of triangle ABC.

2.

a) Prove that $\phi_1 = \phi_2 = \phi$.

b) Find ϕ as a function of the angles of ABC.

Definition 2. Angle ϕ is called the *Brocard angle* of triangle ABC.

3. Prove that the projections of Brocard points to the sidelines of ABC are concyclic. (This is true for any pair of isogonally conjugated points).

4. Let O be the circumcircle of ABC.

a) Prove that $OP = OQ$.

b) Prove that $\angle POQ = 2\phi$.

Definition 3. The reflections of the medians of a triangle in its correspondent bisectors are called the *symmedians*. Three symmedians concur in point L, which is called the *Lemoine point* of the triangle.

5. Prove that P and Q lie on the circle with diameter OL.

6. (K.Knop) Consider two triangles: one of them is formed by the circumcenters of triangles PAB, PBC, PCA; the second one is formed by the circumcenters of triangles QAB, QBC, QCA. Prove that these triangles are

a) similar to ABC;

b) equal.

c) Find the center and the angle of the rotation transforming one of these triangles to the second one.

7. Let C' be a point of segment AB, such that AB is the external bisector of angle $PC'Q$. Prove that CC' is the symmedian of ABC. (I.e. there exists an ellipse with foci P and Q touching the sides of the triangle in the bases of its symmedians).

8. Let T_1, T_2 be points of line OL, such that $\angle LPT_1 = \angle LPT_2 = 60^\circ$. Prove that the projections of each of these points to the sidelines of ABC form a regular triangle (these points are called the *Apollonius points*).

2 Brocard points in quadrilaterals

9. Let $ABCD$ be a convex broken line. Prove that there exists a unique point P, such that $\angle PAB = \angle PBC = \angle PCD = \phi$.

Definition 4. We will call P and ϕ the *Brocard point* and the *Brocard angle* of broken line $ABCD$. We will denote them as $P(ABCD)$ and $\phi(ABCD)$.

10. Find $\phi(ABCD)$ as a function of the lengths of segments AB, BC, CA and the angles between them.

11. Prove that $\phi(ABCD) = \phi(DCBA)$ iff A, B, C, D are concyclic.

Now we will consider only cyclic polygons.

12. Let $P_1 = P(ABCD)$, $P_2 = P(BCDA)$, $P_3 = P(CDAB)$, $P_4 = P(DABC)$. Prove that $P_1P_2P_3P_4$ is a cyclic quadrilateral.

13. Let $Q_1 = P(DCBA)$, $Q_2 = P(ADCB)$, $Q_3 = P(BADC)$, $Q_4 = P(CBDA)$. Prove that $P_1P_2/Q_1Q_2 = BC/CD$, $P_2P_3/Q_2Q_3 = CD/DA$ etc.

14. (D.Belev) Let M_1, M_2 be points on lines AD, AB respectively such that $BM_1 \parallel CD$, $CM_2 \parallel DA$.

a) Prove that the circumcircles of triangles BAM_1 and BCM_2 meet in P_1.

b) Define the similar construction for $P_i, i = 2, \ldots, 4$, $Q_i, i = 1, \ldots, 4$.

15. (D.Belev) Prove that lines CP_1, DP_2, AP_3, BP_4 concur, and lines BQ_1, CQ_2, DQ_3, AQ_4 concur.

16. (D.Belev) Denote the points obtained in the previous problem as P_0, Q_0.

a) Prove that $S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}$

b) Prove that the areas of $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ are equal.

17. Prove that $\phi(ABCD) = \phi(BCDA)$ iff $AB \cdot CD = AD \cdot BC$.

Definition 5. A cyclic quadrilateral with equal products of opposite sides is called *harmonic*. From the last problem we obtain that in the harmonic quadrilateral there exist points P and Q, such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \angle QDC = \angle QCB = \angle QBA = \angle QAD = \phi$. We will call P, Q and ϕ the *Brocard points* and the *Brocard angle* of quadrilateral $ABCD$.

18. Prove that each of the following conditions is true iff $ABCD$ is harmonic.

a) The tangents to the circumcircle in A and C meet on BD.

b) BD is a symmedian of ABC.

c) The distances from the common point L of the diagonals to the sides are proportional to these sides.

d) There exists an inversion transforming A, B, C, D to the vertices of a square.

e) There exists a central projection transforming $ABCD$ and its circumcircle to a square and a circle.

19. Find the Brocard angle of a harmonic quadrilateral as a function of its angles.

20. Prove that $OP = OQ$ and $\angle POQ = 2\phi$.

21. Prove that P and Q lie on the circle with diameter OL.

3 Brocard points in polygons

22. Let a circle, a point P inside it and an angle ϕ be given. For an arbitrary point X_0 on the circle construct a point X_1, such that the oriented angle PX_0X_1 is equal to ϕ. Similarly for X_1 construct X_2 etc. Prove that if $X_n = X_0$, then this is true for any other initial point.

23. Find the closure condition in the previous problem.

Remind that all considered polygons are cyclic.

Definition 6. We will call a polygon $A_1 \dots A_n$ a *Brocard* polygon if there exists a point P, such that $\angle PA_1A_2 = \angle PA_2A_3 = \cdots \angle PA_nA_1 = \phi$.

24. Prove that in a Brocard polygon there exists a point Q such that $\angle QA_1A_n = \angle QA_nA_{n-1} = \cdots \angle QA_2A_1 = \phi$.

Definition 7. We will call P, Q and ϕ the *Brocard points* and the *Brocard angle* of $A_1 \dots A_n$.

25. Prove that each of the following conditions is true iff $A_1 \dots A_n$ is the Brocard polygon.

a) There exists a point L, such that the distances from it to the sides of the polygon are proportional to these sides.

b) The symmedians of triangles $A_1A_2A_3, A_2A_3A_4, \dots, A_nA_1A_2$ from $A_2, A_3, \dots, A_1$ concur.

c) The common points of lines $A_1A_3, A_2A_4, \dots, A_nA_2$ with the tangents to the circumcircle in $A_2, A_3, \dots, A_1$ respectively are collinear.

d) There exists an inversion transforming $A_1, \dots, A_n$ to the vertices of a regular triangle.

e) There exists a central projection transforming the polygon and its circumcircle to a regular polygon and a circle.

26. Prove that the Brocard points lie on the circle with diameter OL and $\angle POL = \angle QOL = \phi$.

27.

a) Prove that there exist two points T_1, T_2 such that the inversion with the center in any of them transforms $A_1, \dots, A_n$ to the vertices of a regular triangle.

b) Prove that T_1, T_2 lie on OL and $\angle T_1PL = \angle T_2PL = \frac{\pi}{n}$.

28. Find the Brocard angle as a function of OL/R.