Part 1

Problem 1-1.

Proof. a) Let a function g be determined on $[-\pi, \pi]$ and periodical with period 2π. Consider the segment $[x_1, x_0], x_1 = 1, y_1 = \frac{\pi}{2}$ and let t_0 be its midpoint. Denote by x_n, t_n the n-th preimages of x_0, t_0 taken in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Suppose $[t_0, x_0] \xrightarrow{g} [x_1, t_0]$ acts on t_0 as a translation. Now let $[x_1, t_0] \xrightarrow{g} [t_1, x_1]$ act so that $g^{(2)} \equiv \sin$ on $[t_1, x_1]$, similarly for $[x_2, t_1]$ and so on. Similarly for $[-\frac{\pi}{2}, 0]$. Let $g(x) = g(\pi - x), \forall x \in [\frac{\pi}{2}, \pi]$ and similarly for $[-\pi, -\frac{\pi}{2}]$. It is easily seen that the resulting function satisfies the conditions of Problem 1-1 and moreover is continuous.
b) Divide all points into classes such that their images under cos are the same. Each class will be represented by a point of $[0, \pi]$. The class containing a point x will be denoted by $[x]$. Now determine the action of the function g on the classes as follows. For the class containing a fixed point x_0 let $g([x_0]) = x_0$. Divide the remaining classes into sets of the form

$$\{\dots, [\cos^{(-1)}(\alpha)], [\alpha], [\cos(\alpha)], [\cos^{(2)}(\alpha)], \dots\}.$$

The divide these sets into pairs, and for the pair generated by classes $[\alpha], [\beta]$ determine the following action: $g([\cos^{(n)}(\alpha)]) = g([\cos^{(n)}(\beta)]), g([\cos^{(n)}(\beta)]) = g([\cos^{(n+1)}(\alpha)])$. Clearly the square of the resulting function is cos. □

Solutions of **problems 1-2, 1-3, 1-4** are contained in a paper of V. Vikola and A. Apostolov, see "Matematicheskoe prosveschenie", year 2004, vol. 9.

Problem 1-5.

Proof. Let us consider what are possible forms of the substitutions.

1. The identical substitution. It is its own square.
2. A transposition. Its square is the identical substitution.
3. A product of two disjoint transpositions. Its square is the identical substitution.
4. A cycle of length 3. Its square is the cycle of length 3 inverse to the original one.
5. A cycle of length 4. Its square is a product of disjoint substitutions.

In total, we get $1 + 3 + 8$ substitutions which are squares of substitutions. In the case of 9 elements, a similar argument implies that the cubes are any substitutions without cycles of length 6 and with 0 or 3 cycles of length 3. $\square$

Part 2

Problem 2-1.

Proof. a) Let $f(x) = x + c$. Then $f^{(\lambda)}(x) = x + \lambda c$.
b) Let $f(x) = \alpha x$. Then $f^{(\lambda)}(x) = \alpha^\lambda x$.
c) Let $f(x) = \alpha x + c$. We assume that $\alpha \neq 1$ (cf. a). Then

$$f(x) = \alpha\left(x - \frac{c}{1-a}\right) + \frac{c}{1-\alpha}.$$

Therefore

$$f^{(\lambda)}(x) = \alpha^\lambda\left(x - \frac{c}{1-\alpha}\right) + \frac{c}{1-\alpha}.$$

d) Let $f(x) = ax^n$. We assume that $a \neq 1$ (cf. b). The solution of this problem is similar to part c and this is not an occasion.

$$f(x) = \sqrt[1-n]{c}(x/\sqrt[1-n]{c})^n.$$

$$f^{(\lambda)}(x) = \sqrt[1-n]{c}(x/\sqrt[1-n]{c})^{\lambda n}.$$

$\square$

Problem 2-2.

Proof. Let $f(x) := x^2 - 2$. We substitute x in f by $u + \frac{1}{u}$. We have

$$f(u + \frac{1}{u}) = u^2 + \frac{1}{u^2}.$$

Further

$$f^{(\lambda)}(u + \frac{1}{u}) = u^{2^\lambda} + \frac{1}{u^{2^\lambda}}.$$

Then

$$f^{(\lambda)}(x) = \left(\frac{x + \sqrt{x^2-1}}{2}\right)^{2^\lambda} + \left(\frac{x - \sqrt{x^2-1}}{2}\right)^{2^\lambda}.$$

$\square$

Problem 2-3-a.

Proof. Set $z = u + \frac{1}{u}$. As $z \notin \mathbb{R}$, $|u| \neq 1$. By the previous theorem

$$f^{(n)}(z) = f^{(n)}(u + \frac{1}{u}) = u^{2^n} + \frac{1}{u^{2^n}}.$$

We have

$$\lim_{n\to\infty} |u^{2^n} + \frac{1}{u^{2^n}}| = +\infty.$$

□

Problem 2-3-b.

Proof. Let $f(x)$ be a polynom of degree n with the highest coefficient 1. Let $x_1, ..., x_n$ be roots of f. The coefficients

$$f_n(x) := (x - x_1)...(x - x_n)$$

are symmetric polynomials of $x_1, ..., x_n$ with integer coefficients. Therefore all coefficients of f_n are polynoms of the elementary symmetric polynoms (:=coefficients of f). Therefore all coefficients of f_n are integers. As $|x_i| = 1$ for all i, $|x_i^k| = 1$ for all i and k. Therefore for any k the coefficients of f is smaller or equal to $n!$. Hence there exist different numbers k_1, k_2 such that $f_{k_1} = f_{k_2}$. But then the set of numbers

$$\{x_1^{k_1}, ..., x_n^{k_1}\} \text{ and } \{x_1^{k_1}, ..., x_n^{k_1}\}$$

coincide. Therefore x_i-th are roots of unity (see also "Matematicheskoe Prosveschenie", year 2004, vol. 9). □

This problem was invented by M. Kontsevich.

Problem 2-3-c.

Proof. Let n be a degree of $P(x)$ and $\{x_1, ..., x_n\}$ be roots of $P(x)$. Then the roots of

$$u^n P(u + \frac{1}{u})$$

are solutions of equality $u + \frac{1}{u} = x_i$. As $x_\in[-1.99, 1.99]$, all roots are purely complex and has length 1. Therefore polynom $u^n P(u + \frac{1}{u})$ has integer coefficients, and all roots of this polynom are roots of unity by Problem **2-3-c**. Let u_i be root of unity of degree m_i, such that u_i is a root of $P(x)$. Then u_i^r

is a root of polynom $u^n P(u + \frac{1}{u})$ for any integer r such that $(r, m_i) = 1$ (see also ¡¡ ¿¿, year 2004, vol. 9).

There exists N such that $m > N$ and for any primitive root of unity u of degree m there exists number r, such that $(m, r) = 1$ and $2\mathrm{Re} u^r > 1.99$. Therefore there exists a finite set $\mathcal{S}$ such that any root of a polynom $P(x)$, which satisfies the conditions of the problem, lies in $\mathcal{S}$. □

Part 3

Problem 3-1.

Proof. Let K be a big map. Introduce the map $f : K \to K$ that maps each point of the big map to its point lying under the corresponding point of the small map. Then the problem reformulates as follows: find a fixed point for f. Clearly $f(K)$ is a rectangle in K, $f(f(K))$ is a rectangle in $f(k)$ and so on. We obtain a sequence of similar rectangles embedded into each other, such that their greater sides form a geometrical progression whose denominator is less than 1. These rectangles have a single common point x. Suppose $a \neq 0$ is the distance between x and $f(x)$. There exists a rectangle $f^{(m)}(K)$ whose diagonal has length less than a. Since $f(x)$ lies inside $f(f^{(m)}(K))$ and $f^{(m+1)}(K)$ lies inside $f^{(m)}(K)$, the distance between x and $f(x)$ is less than a, a contradiction. Thus $x = f(x)$. □

Problem 3-2.

Proof. Consider the function $f(x) = \sqrt{1+x}$. Our sequence has the form $x_n := f^{(n)}(1)$. Solving the equation $f(x) = x$ we see that $\frac{1+\sqrt{5}}{2}$ is a single fixed point of f. Furthermore for $0 < x < \frac{1+\sqrt{5}}{2}$ we have the double inequality $x < f(x) < \frac{1+\sqrt{5}}{2}$. Hence the sequence x_n is strictly increasing and bounded and hence has some limit h. Since f is continuous, the sequence $f(x_n)$ tends to $f(h)$, whence $f(h) = h$. Thus $h = \frac{1+\sqrt{5}}{2}$. □

Problem 3-3.

Proof. Denote $f(x) := sin(tg(x))$, $g(x) := tg(sin(x))$. We have to find the limit

$$\lim_{x \to 0} \frac{f(x) - g(x)}{g^{-1}(x) - f^{-1}(x)}.$$

Observe that f, g, f^{-1} and g^{-1} are infinitely differentiable in some neighborhood of zero and fix 0. Now observe that $\lim_{x\to 0}\frac{f(x)}{x} = 1$. The same is true for g, f^{-1} and g^{-1}.

Suppose that for $h(x) = g^{-1}(x) - f^{-1}(x)$ first $k-1$ derivatives are zeroes at zero and the kth derivative is not. Then

$$\lim_{x\to 0}\frac{g^{-1}(x) - f^{-1}(x)}{x^k} = \frac{h\overbrace{'' \cdots '}^{k \text{ times}}(0)}{k!}.$$

Substitute $f(x)$ for x. We get

$$\lim_{x\to 0}\frac{g^{-1}(f(x)) - f^{-1}(f(x))}{(f(x))^k} = \frac{h\overbrace{'' \cdots '}^{k \text{ times}}(0)}{k!}.$$

Hence

$$\lim_{x\to 0}\frac{g^{-1}(x) - f^{-1}(x)}{g^{-1}(f(x)) - x} = 1.$$

From $\lim_{x\to 0} f^{-1}(x) = 1$ we get

$$\lim_{x\to 0}\frac{f^{-1}(f(x)) - f^{-1}(g(x))}{f(x) - g(x)} = 1.$$

Let $s(x) := f^{-1}(g(x))$. Then the problem reduces to finding the limit

$$\lim_{x\to 0}\frac{x - s(x)}{s^{-1}(x) - x}.$$

If $s(x)$ in a neighborhood of zero decomposes to the Taylor series of the form $x + ax^k + \ldots$ then the Taylor series for $s^{-1}(x)$ has the form $x - ax^k + \ldots$. Thus the limit of the ratio of the numerator and the denominator equals 1. $\square$

Part 4

Problem 4-1.

Proof. We introduce a coordinate system such that A has coordinates $(0,0)$, point B – (0, 1), point C – (1, 0). Let X be a point of coordinates (a,b). Then after an "attraction" to one of the points A, B, C we have points with coordinates.

$$\left(\frac{a}{2},\frac{b}{2}\right), \qquad \left(\frac{a}{2}+\frac{1}{2},\frac{b}{2}\right), \qquad \left(\frac{a}{2},\frac{b}{2}+\frac{1}{2}\right)$$

Hence, for any $\varepsilon > 0$ there exists n such that after several "attractions" both coordinates a and b becomes bigger then $-\varepsilon$. Therefore after several attractions distance between any point and $\triangle ABC$ becomes smaller than ε. Without loss of generality we assume that $x \in \triangle ABC$.

Attraction to A of a set $\mathcal{X} \subset \triangle ABC$ produces a set $\mathcal{X}_A$, which is 4 times smaller by area. It is easy to see that area of

$$\mathcal{X}_A \cup \mathcal{X}_B \cup \mathcal{X}_C$$

is smaller than 3/4 of area of $\mathcal{X}$ or is equal to it. Thus after n-th attraction any point of ABC comes to union of triangles (¡¡Serpinski's curpet¿¿), the common area of which is $(3/4)^n$ of ABC or smaller. Therefore there exists a figure $\mathcal{F}$ with area 0.00000001 such that any sequence of attractions of any point hits $\mathcal{F}$. □

Problem 4-2.

Proof. Denote $f(x) = x^2 - 10$. The equation $f(x) = x$ has exactly two real solutions, namely $x_{1,2} = \frac{1\pm\sqrt{41}}{2}$. Note that x_1, x_2 are fixed points of the map $x \to f(x)$. For certainty assume $x_2 > x_1$. Let x_2^* be the root of $f(x) = x_2$ distinct from x_2, and let y_1, y_2 be roots of $f(x) = x_2^*$ such that $y_1 < y_2$. Observe that f is monotonous at segments $[x_2^*, y_1]$ and $[y_2, x_2]$ and the images of these segments coincide with $[x_2^*, x_2]$. It is important that the derivative of f at $[x_2^*, y_1]$ and $[y_2, x_2]$ is not less than 3.

For all $x \in [-10, x_2^*] \cup [x_2, +\infty)$ the sequence $x, f(x), f(f(x)), \ldots$ tends to infinity and increases starting with the second term. Hence the set of points x such that the limit for $x, f(x), f(f(x)), \ldots$ does not exist or is not infinite, coincides with

$$\cap_i f^{(-i)}([x_2^*, x_2]) = \cap_i f^{(-i)}([x_2^*, y_1] \cup [y_2, x_2]).$$

The preimage $f^{(-i)}([x_2^*, y_1] \cup [y_2, x_2])$ consists of 2^{i+1} pieces, and each piece maps bijectively to some piece $f^{(-i+1)}([x_2^*, y_1] \cup [y_2, x_2])$. Since the derivative of f is not less than 3 on $[x_2^*, y_1] \cup [y_2, x_2]$, the total length $f^{(-i)}([x_2^*, y_1] \cup [y_2, x_2])$ does not exceed $2/3$ of the total length of $f^{(-i+1)}([x_2^*, y_1] \cup [y_2, x_2])$. Thus it is evident that $f^{(-i)}([x_2^*, y_1] \cup [y_2, x_2]) < 0.0000001$ for some i. Hence $f^{(-i)}([x_2^*, y_1] \cup [y_2, x_2]) < 0.0000001$ is the required system of segments that contains all points x such that the limit of $x, f(x), f(f(x)), \ldots$ does not exist or is not infinity. □

Part 5

Problem 5-1.

Proof. Let us find the derivative of

$$(f^{(n)}(x) - x)' = -\sin(f^{(n-1)}) \cdot (f^{(n-1)})' - 1.$$

Observe that further similar decomposition of the derivative shows that the first term has the absolute value not greater than 1. Hence the function $f^{(n)}(x) - x$ monotonically decreases. Furthermore the zero clearly is a solution for $\cos x = x$, and this is the only solution for the original equation. □

Problem 5-2.

Proof. Let us start with solving of the equation $f(x) = x$, that is, $1 - x^2 = x \Leftrightarrow x^2 + x - 1 = 0$. Its solutions are $x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$. Furthermore for $x < \frac{-1-\sqrt{5}}{2}$ the values of $f^{(n)}(x)$ decrease. The image of $[-1, 0]$ is the segment $[0, 1]$, hence the equation has no solutions on this segment. Similarly we may take its preimage, the preimage of its preimage and so on to obtain that the equation has no solutions on $[\frac{-1-\sqrt{5}}{2}, 0]$. Now denote $\frac{-1+\sqrt{5}}{2}$ by φ and observe that $\forall x \in [0, \varphi]$ $f(x) \in [\varphi, 1]$ and $\forall x \in [\varphi, 1]$ $f(x) \in [0, \varphi]$. Thus for iterations of f with odd numbers the only roots are $x_{1,2}$. Now consider $f^{(2)}$. It is determined by the polynomial $2x^2 - x^4$ of degree 4. The function $f^{(2)}(x) - x$ has four roots, two of which are $x_{1,2}$ and two others are 0 and 1. This implies that on $[0, \varphi]$ the graph of $f^{(2)}$ is located on one side of line $y = x$, and iterations $f^{(2)}$ that are iterations of f with even numbers will change the value of $f^{(2k)}(x)$ monotonously inside the interval $(0, \varphi)$. Hence this interval contains no solutions. Similarly for $(\varphi, 1)$. Thus there exist two solutions $\frac{-1 \pm \sqrt{5}}{2}$ for all n and two more solutions 0 and 1 for even n. □

Problem 5-3-b.

Proof. Let f, g be a pair of functions such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

If $\lim_{x\to 0} \frac{g(x)}{f(x)} = 0$ then we will allow notation $o(f(x))$ for $g(x)$. We will prove a statement more general than that of Problem 5-3-b.

Statement. If f$(x) = x - \frac{x^3}{6} + o(x^3)$ then $\lim\limits_{n\to\infty} \sqrt{n} f^{(n)}(x_0) = \sqrt{3}$ for all x_0 sufficiently close to 0. To prove this, denote the function $\sqrt{x}$ by $SQ(x)$. Consider the function

$$\tilde{f} := SQ^{-1} \circ f \circ SQ.$$

It is easy to see that $(SQ^{-1} \circ f \circ SQ)(x) = x + \frac{1}{3} + o(1)$.

Proposition. Suppose for some function $\tilde{f}$ we have

$$\tilde{f}(x) = x + \frac{1}{3} + 0(1).$$

Then for all x with sufficiently small absolute value we have $\lim\limits_{n\to\infty} \frac{\tilde{f}^{(n)}(x)}{n} = \frac{1}{3}$.

Proof. For any x with sufficiently small absolute value and for all $\varepsilon > 0$ there exists N such that for any $n > N$ we have

$$f^{(n+1)}(x) \in [f^{(n)}(x) + \frac{1}{3} - \varepsilon, f^{(n)}(x) + \frac{1}{3} - \varepsilon].$$

Hence for any $n > N$ we have

$$f^{(n)}(x) \in [f^{(N)}(x) + \frac{1}{3}(n-N) + (n-N)\varepsilon, f^{(N)}(x) + \frac{1}{3}(n-N) - (n-N)\varepsilon]$$

and thus

$$\frac{1}{3} - \varepsilon + \frac{f^{(n)}(x)}{n-N} < \frac{f^{(n)}(x)}{n-N} < \frac{1}{3} - \varepsilon + \frac{f^{(n)}(x)}{n-N}.$$

Consequently there exists N' such that for all $n > N'$ we have

$$\frac{1}{3} - 2\varepsilon < \frac{f^{(n)}(x)}{n} < \frac{1}{3} + 2\varepsilon.$$

Hence $\lim\limits_{n\to\infty} \frac{f^{(n)}(x)}{n} = \frac{1}{3}$. $\square$

Suppose $x = SQ(y)$. Note that

$$SQ^{(-1)}(f^{(n)}(x)) = SQ^{(-1)}(f^{(n)}(SQ(y))) = (SQ^{(-1)} \circ f \circ SQ)^{(n)}(y).$$

Remind that $\tilde{f} := SQ^{(-1)} \circ f \circ SQ$. By the above Proposition, $\lim\limits_{n\to\infty} \frac{\tilde{f}^{(n)}(y)}{n} = \frac{1}{3}$. Hence

$$\lim_{n\to\infty} \frac{(\frac{1}{f^{(n)}(x)})^2}{n} = \frac{1}{3}.$$

Thus $\lim\limits_{n\to\infty} \sqrt{n} f^{(n)}(x) = \sqrt{3}$. $\square$

For conjugating function (SQ) we can use the function $\mathrm{e}^{-\frac{1}{x^2}}$ as well. Investigate this case by yourself.

Problem 5-3-4. The solution of this problem coincides with that of Problem 5-3-2 but we find it worth while to repeat it.

Proof. We will prove a statement more general than that of Problem 5-3-2.

Statement. If $\mathrm{f}(x) = x - ax^k + o(x^k)$ then

$$\lim_{n\to\infty} \sqrt[k-1]{n} f^{(n)}(x) = \sqrt[k-1]{\frac{1}{(k-1)a}}$$

for all x sufficiently close to 0.

Denote the function $\sqrt[k-1]{x}$ by $SQ(x)$ and consider the function

$$\tilde{f} := SQ^{-1} \circ f \circ SQ.$$

It is easy to see that $(SQ^{-1} \circ f \circ SQ)(x) = x + (k-1)a + o(1)$.

Statement. Suppose for some function $\tilde{f}$ we have

$$\tilde{f}(x) = x + \alpha + 0(1).$$

Then for all x with sufficiently small absolute value we have $\lim\limits_{n\to\infty} \frac{\tilde{f}^{(n)}(x)}{n} = \alpha$.

Suppose $x = SQ(y)$. Note that

$$SQ^{(-1)}(f^{(n)}(x)) = SQ^{(-1)}(f^{(n)}(SQ(y))) = (SQ^{(-1)} \circ f \circ SQ)^{(n)}(y).$$

Remind that $\tilde{f} := SQ^{(-1)} \circ f \circ SQ$. By the above Proposition $\lim\limits_{n\to\infty} \frac{\tilde{f}^{(n)}(y)}{n} = (k-1)a$. Hence

$$\lim_{n\to\infty} \frac{(\frac{1}{f^{(n)}(x)})^{(k-1)}}{n} = \sqrt[k-1]{\frac{1}{(k-1)a}}.$$

Thus $\lim\limits_{n\to\infty} \sqrt[k-1]{n} f^{(n)}(x) = \sqrt[k-1]{\frac{1}{(k-1)a}}$. $\square$

Problem 5-5.

Proof. Let $f(x) = x - exp(-1/x^2)$, $x_n = f^{(n)}(x)$ and let $y_n = 1/x_n$. Then we have $y_{n+1} - y_n \sim y_n \exp(y_n^{-2})$.

Change our equation with a differentiable because it will not change asymptotics. Let estimate function $f(n)$ for which $y' = ye^{-y^2}$, $dy/ye^{y^2} = 1$, or $\int_1^h \exp(u^2)/u^2 du^2 = 2h + C$. We can see that for sufficiently large y y^2 is constant comparing to $\exp(y^2)$, From asymptotic point of view we can change function to z such that

$\frac{1}{2h^2}\int_1^h du^2 e^{u^2} \sim t$, so $\frac{e^{h^2}}{2h^2} \sim n$

or

$h^2 \sim \ln(n) + 2\ln(h) \sim \ln(n) + \ln(\ln(n)) \quad h \sim \sqrt{\ln(n)}$.

Hence $\lim \sqrt{\ln(n)} \cdot x_n = 1$.

□

Part 6

Problem 6-1.

Proof. Let $f(x)$ commute with $y(x)$. Then $2f(x) = f(2x)$. Note that

$$f(1) = 2f(\frac{1}{2}) = 4f(\frac{1}{4}) = ...,$$

and so $f(0) = 2f(0)$, hence $f(0) = 0$. Let us show that $\frac{f(x)}{x}$ is a constant. In fact,

$$\frac{f(x)}{x} = \frac{f(\frac{x}{2})}{\frac{x}{2}} = \frac{f(\frac{x}{4})}{\frac{x}{4}} = ... = f'(0)$$

for any x. Hence $f(x) = f'(0)x$. □

Problem 6-2.

Proof. Consider an arbitrary continuous function $f_{2,4}$ defined on $[2, 4]$ and equal to 0 at its endpoints. By formulas

$$f(2x) = 2f(x), f(-x) = f(x), f(0) = 0$$

extend $f_{2,4}$ to the whole real axis. Obviously the resulting function is defined and differentiable at all points including zero. If $f_{2,4}$ is not differentiable in some internal point thenobviously the same holds for f. □

Part 7

Problem 7-1

Proof. a) We will prove this problem using mathematical induction.
The base obviously holds.

Inductive step: $n \to n+1$.

$$\begin{gathered}\cos(n \cdot \arccos(x)) = \\ = x\cos((n-1)\cdot\arccos(x)) - \sin(\arccos(x))\sin((n-1)\arccos(x)) = \\ = x^2\cos((n-2)\cdot\arccos(x)) - \sin^2(\arccos(x))\sin((n-2)\arccos(x)) - \\ 2x\sin(\arccos(x))\sin((n-2)\arccos(x)) = \\ = x^2\cos((n-2)\cdot\arccos(x)) - (1-\cos^2(\arccos(x)))\cos((n-2)\arccos(x)) - \\ 2x\sin(\arccos(x))\sin((n-2)\arccos(x)).\end{gathered}$$

It is clear that the first and the second summand of this sum are polynomials. As

$$\begin{gathered}2x\sin(\arccos(x))\sin((n-2)\arccos(x)) = \\ = 2x\cos(\arccos(x))\cos((n-2)\arccos(x)) - 2x\cos((n-1)\arccos(x)),\end{gathered}$$

, $\cos(n\arccos(x))$ is a polynom.
b) We have

$$\begin{gathered}\sin((2n-1)\arcsin(x)) = \sin((2n-1)(\tfrac{\pi}{2} - \arccos(x))) = \\ = \sin(\tfrac{(2n-1)\pi}{2} - (2n-1)\arccos(x))) = \pm\cos((2n-1)\arccos(x)),\end{gathered}$$

Therefore $\sin((2n-1)\arcsin(x))$.
) We will prove this problem using mathematical induction. The base obviously holds.

Inductive step: $n \to n+1$.

$n \to n+1$.

$$\tan(n\arctan(x)) = \frac{\tan((n-1)\arctan(x)) + x}{1 - x\tan((n-1)\arctan(x))}.$$

In this fraction the numerator and the denumerator are rational functions. Therefore the quotient of them is a rational function too. □

Problem 7-2.

Proof. The set of roots of $\sin(x) = 0$ is denumerable. From the other hand the number of roots of $P(x) = a$ is always countable for any a. Therefore $\sin(x)$ is not conjugated to a polynom. □

Problem 7-3

Proof. We assume that $c \neq 0$ (cf. 2-1-c). If we find a function R such that $R \circ f \circ R^{(-1)}$ is a linear function. Hence we know all fractional iterations of linear function, such function R provide a solution of problem **7-3**. First we conjugate f by function $R_1 = x + \frac{a}{c}$. We have

$$f_1 := R_1 \circ f \circ R_1^{(-1)} = \alpha + \frac{\beta}{x},$$

where α and β are some numbers. Then we conjugate f_1 by $R_2 := x\sqrt{\beta})$. We have

$$f_2 = R_2 \circ f_1 \circ R_2^{(-1)} s + \frac{1}{x}$$

. Let λ be a number such that $\lambda(\lambda - s) - 1 = 0$. We conjugate f_2 by $R_3 := 1 + \frac{1}{x+\lambda}$. We have

$$f_3 := R_2 \circ f \circ R_2^{(-1)}.$$

The function f_3 is linear. □

Problem 7-4.

Proof. Note that

$$(f^n)'(0) = f'(f(f(\cdots)))(0) \cdot f'(f(\cdots))(0) \cdots f'(0) = k^n.$$

Then

$$(R \circ f \circ R^{(-1)})'(0) = R'(f(R^{(-1)}(0)))f'(R^{(-1)}(0))(R(-1))'(0) =$$

$$= f'(0) \cdot R'(0) \cdot (R(-1))'(0) = f'(0).$$

We recall that $|f'(0)| = |k| < 1$. Let q be a real number such that $q : |k| < q < 1$. As $|f'(0)| < q$, there exists a neighborhood of 0 such that for any point from this neighborhood $|f(x)| < q\,|x|$. Therefore for any point x of this neighborhood

$$|f^n(x)| < q^n x,$$

and hence $f^{(n)}(x) \rightrightarrows 0$ for $n \to \infty$. □

Problem 7-5.

Proof. We'll start our proof with some useful facts.

Proposition. Let f be a twice differentiable function such that $f(0) = 0$ and $f'(0) = k$, for $0 < k < 1$. Then

$$\exists C > 0 \exists \varepsilon > 0 \mid \forall x \in (-\varepsilon, \varepsilon)\ |f^{(n)}(x)'| < Ck^n.$$

Proof. Let f, g be functions such that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0.$$

If function $\frac{g(x)}{f(x)}$ exists and is bounded in 0 neighborhood, we will write $O(f(x))$ instead of $g(x)$.

Let us find $f^{(n)}(x) =$

$$f'(x) f'(f(x)) ... f'(f^{(n-1)} x).$$

By

$$f'(f^{(m)}(x)) = k(1 + O(f^{(m)}(x))) =$$

we conclude that

$$\frac{f^{(n)\prime}(x)}{k^n} = (1 + O(x))(1 + O(f(x))) ... (1 + O(f^{(n-1)}(x))).$$

Series

$$x + f(x) + f(f(x)) + ...$$

is absolutely convergent, there exists constant number C such that $f^{(n)}(x) < Ck^n$. $\square$

For function f and integer m we will write $f^{[m]}$ for m-th derivative of f.

Proposition. Let f be a function such that $f(0) = 0$ and $f'(0) = k$, for $-1 < k < 1$. Then

$$\forall m \geq 1 \exists C_m > 0 \exists \varepsilon_m > 0 \mid \forall x \in (-\varepsilon_m, \varepsilon_m) |f^{(n)}(x)^{[m]}| < C_m k^n.$$

Proof. We will use the following lemma:

Proposition. Let exist a constant C for some $m > 0$ that, for each $r \leq m$ we have $f^{(n)}(x)^{[m]} \leq Ck^n$, then for each r such that $1 < r \leq m$ we have $(f'(f^{(n)}(x)))^{[r]} \leq C' k^n$ for some constant C'.

Proof. When we differentiate $f'(f^{(n)}(x))$ at most m times in each summand has $f^{(n)}$ or derivative $f^{(n)}$ with order not more than m. $\square$

Let us recall $f^{(n)}(x)' =$

$$f'(x)f'(f(x))...f'(f^{(n-1)}(x)).$$

Note that $f^{(n-1)}(x)^{[m]} = (f^{(n-1)}(x)^{[1]})^{[m-1]}$. We will prove the statement using induction. Base is a previous Proposition. Inductive step $m \to m+1$. Expression $f^{(n)}(x)'$ is a product of n parts and taking its m-th derivative is a sum of n^m summands, each having no more than m factors $f^{(n)}(x)'$ changed. Consider all expressions with exactly $r \leq m$ factors changed. Derivative of changed factor $f'(f^{(s)})(x)$ is less or equal to $C'k^s$, sum of changed factors is less or equal to

$$m!k^{n-r} \sum_{i_1,...,i_r \leq n} \prod_{q=1}^{r} (C'k^{i_q}) = m!C'^r k^{n-r} \prod_{q=1}^{r} (1+...+k^n) \leq m!C'^r k^{n-r} \left(\frac{1}{1-k}\right)^r.$$

Obviously, summing those for all r is less or equal to $C''k^n$ for some $C'' > 0$. $\square$

If for a set $\mathcal{F}$ of continiously differentiable functions there exists C such that $\forall f \in \mathcal{F}$ we have

$$|f(x)| < C \text{ and } |f'(x)| < C$$

for each x, then there exists a sequence of distinct functions $f_1, f_2, ..., f_n$ from $\mathcal{F}$ ([**?**]Z) Arzel-Ascoli's theorem). From evenly boundeness of m-th derivatives of $\frac{f^{(n)}}{k^n}$ for each $m \geq 1$, there exists a sequence $n_1, n_2, ...$ such that sequence $\frac{f^{(n_1)}}{k^{n_1}}, \frac{f^{(n_2)}}{k^{n_2}}, ...$ converges evenly with series of its m-th derivatives for all $m \geq 1$. Hence, sequence

$$\frac{f^{(n_1)}}{k^{n_1}}, \frac{f^{(n_2)}}{k^{n_2}}, ...$$

converges evenly, hence, it converges to G, which is the limit of this sequence at each point, from which we conclude that G is smooth. From the equality

$$\frac{f^{(n-1)}(f(x))}{k^{n-1}} = k\frac{f^{(n)}(x)}{k^n}$$

we have

$$G(f(x)) = kG(x).$$

Derivative at 0 of all functions in sequence $\frac{f^{(n)}(x)}{k^n}$ equals 1, so derivative of $G(x)$ at 0 is 1. Hence, G is invertible in 0 neighborhood $f(x) = G^{(-1)}(kG(x))$. □

Remark. Instead of Arzel-Ascoli theorem we can use the following fact: for a sequence of functions $\{f_i\}_{i\in\mathbb{Z}_{\geq 0}}$ converges and their second derivatives are uniformly bounded, then sequence of first derivatives converges too.

Problem 7-6.

Proof. We set $\frac{x}{2} = \cos(t)$. Then the left side can be written is this way:

$$\prod_{k=1}^{n} \cos(\frac{t}{2^k}).$$

We multiply and divide it by $\frac{2^n}{t}\sin(\frac{t}{2^n})$, which is earning to 1 when $n \to \infty$. We have $\prod_{k=1}^{n} \cos(\frac{t}{2^k}) \sim \frac{\cos(2t)}{t}$. After changing back x to $2\arccos t$ we solve the problem. This problem can also be solved by using problem **7-5**.

Problems **7-8** and **7-9** immediately follows from the problems **7-1**–**7-5**. From the problems **7-1**–**7-9** one can easily get solutions for problems **8-2**. □

Problem 7-10.

Scetch of a proof. We set $f(x) := x - \mathrm{e}^{-\frac{1}{x^2}} (x \geq 0)$. For any $x > 0$ we have $f(x) < x$. The only fixed point of f is 0. We define points $\{x_i\}$ in such way that

$$f(x_i) = x_{i+1}, \qquad x_0 = 1.$$

Let $\delta(x)$ be a smooth function on $[x_0, x_1]$, which has zero derivatives of all orders at x_0 and x_1. Function $f(x)+\delta(x)$ is extendable to the smooth function on $\mathbb{R}_{>0}$. As any derivative of $\mathrm{e}^{-\frac{1}{x^2}}$ equals 0, the extension would be smooth at 0. □

Part 8

Problem 8-1.

Proof. Let $Q(x)$ be a polynom of degree m, such that $Q(x)$ commutes with P(x). Let $q_1, \dots, q_{m-1}$ be roots of $Q'(x)$. Then the polynom $Q(x) - Q(q_i)$ has multiple roots for all i. Therefore the polynom

$$Q(x) - Q(P^{(r)}(q_i))$$

has multiple roots for any integer i and any integer r. Hence the set of numbers $\{Q(P^{(n)}(q_i))\}$ is finite and has not more than m elements. Therefore there exists a finite set $\mathcal{S}$ such that for any polynom Q which satisfies the conditions of the problem the roots of $Q'(x)$ are contained in $\mathcal{S}$.

Hence it is enough to show that there exists only finite set of polynoms $\mathcal{Q}$, such that roots of Q' lies in $\mathcal{S}$ and $Q(x)$ commutes with $P(x)$. Such functions expressed as $\alpha Q_0 + \beta$, where Q_0 depends only on the set of roots of Q' and α, β are some numbers. The set of possible coefficients α is finite because the highest coefficients of

$$\alpha Q_0(P(x)) + \beta = P(\alpha Q_0(x) + \beta)$$

have to coincide. Hence the equality

$$\alpha Q_0(P(0)) + \beta = P(\alpha Q_0(0) + \beta)$$

provides only finite number of possibilities for β. □

Problem 8-2 follows from problems **7-1**–**7-9**.

References

[Z] Vladimir Zorich, Mathematical Analysis I,II, UT, Springer, 2004.