Part D.

1. Triangle $A''B''C''$ is autopolar because its vertices are he common points of quadrilateral $A_1B_1C_1F'$. Since a polar A_1B_1 of point C passes through C'', a polar $A''B''$ of C'' passes through C. Similarly for remaining vertices.

2. Let X be a common point of AA_1 and $A''C''$. The cross-ratio of A, B, C_1 and the common point of AB and A_1B_1 is equal to -1. Projecting these points from C'' to AA_1 we obtain that A_1 is the midpoint of AX.

3. The homothety with center A and coefficient 2 transforms line C_0A_1 to line $A''C''$. Thus these lines are parallel. Using the homothety with center B transforming CC_1 to A_1F' we obtain that the midpoint of $A'A_1$ lies on AB. Thus BA_1C_0A'' is a parallelogram.

4. By previous item C' lies on the medial line of ABC parallel to AB. Then $C'' = C'$. Similarly $A'' = A'$. Thus $F' = F$

5. By problem 3 C_0A_1 is parallel to C_1A_0, i.e. $BC_1 \cdot BA_1 = BC_0 \cdot BA_0$. Expressing these lengths through a, b, c and multiplying to 4 we obtain the sought equality.

6. Given equality yields that C_0A_1 is parallel to C_1A_0. Construct parallelograms BC_1A_0C'' and BA_1C_0A''. Line $A''C''$ passes through B. The homothety with center A and coefficient 2 transforms line C_0A_1 to $A''C''$. Let it transform A_1 to point X. Projecting from C'' points A, X, A_1 and infinity point of AA_1 to line AB we obtain that the cross-ratio of A, B, C_1 and the common point of AB and A_1C'' is equal to -1. Thus A_1C'' passes through B_1. Then $C'' = C'$ i.e. triangles BC_1C' and BAX have a common median. Therefore they are homothetic with center B. So C_1C' is parallel to AA_1 and GC_1FA_1 is a parallelogram. Now our condition follows from an angles calculation.

7. By problem 3 $C'F$ passes through the midpoint of BA_0. Using the homothety with center C' we obtain that $C'F$ passes through the midpoint of $A'B_0$, i.e. $C'F$ is the median of triangle $A'B_0C'$. Similarly for $A'F$.

8. By previous item B_0F is the median of triangle $A'B_0C'$

9. Since O_B is the circumcenter of triangle $A_0B_0C_0$ we obtain that $O_BC_0A_1C'$ is a parallelogram, and by problem 2 BA_1C_0A' is a parallelogram. Thus $A'B = C_0A_1 = O_BC'$.

10. Line SM as Gauss line of $A_1B_1C_1F$ passes through the midpoint of FB_1. Thus as medial line of triangle FGB_1 it is parallel to BB_1.

11. By previous item SM is parallel to O_BF. Then F and O_B lie on the reflection of line O_BF in SM, because they are the reflection of G in M and the reflection of B in S.

12. Since S is the midpoint of BO_B and F divides B_0S in ratio $2:1$ we obtain that F is the centroid of triangle B_0O_BB. Then the median of this triangle from O_B also is divided by F in ratio $2:1$. Thus the endpoint of this median coincide with the midpoint of BB_0, and so with the midpoint of C_0A_0. This yields the assertion of the problem.

13. Consider the polar transformation. Since B' lies on C_0A_0 we obtain that L_B lies on $A'C'$. Now we hawe to prove that the polars of G and M meet on the medial line. Note that the polar of G is parallel to BB' and passes through the common point R of $A_1C_1$and AC. The polar of M passes through B and is parallel to A_1C_1. Points B, B', R and the common point of the polars form a parallelogram. Since the midpoint of BR lies on A_0C_0 and B' also lies on this line, then A_0C_0 passes through the common point of the polars of G and M.

14. Since C_B lies on A_0C_0, L_B lies on its polar, i.e. line AC_A, thus $AL_B \perp CI$. Similarly $AI \perp CL_B$ and we obtain the sought assertion.

15. By angles calculation we obtain that the reflections of vertices of Gergonne triangle in the corresponding bisectors form the triangle homothetic to the original triangle. Thus the lines joining the corresponding vertices of these triangles pass through the homothety center of the incircle and the circumcircle. But G' is the common point of these lines.

16. Clearly A, G, I, G' and C lie on the circle with center on the bisector of angle B, and G' lies inside the triangle. Also G' lies on the line symmetric to BG wrt the bisector of B. This yields the assertion of the problem.

17. Clearly follows from three homothety centers theorem for the incircle, the circumcircle and the Euler circle.

18. Since A_1 and C_1 are symmetric wrt the bisector of angle B, and A_1FC_1G is a parallelogram we obtain using two previous items that A_1FC_1G' is a delthoid with diagonal FM. Thus FM is perpendicular to A_1C_1 and parallel to the bisector of angle B.

19. In next item we will prove that O_B is the midpoint of BL_B. Therefore O_BF is a medial line of triangle L_BBG. Thus F is the midpoint of L_BG. Then F divide L_BE in ratio $2:1$.

20. AL_B is parallel to C_0O_B as two perpendiculars to CI. Thus the homothety with center B and coefficient 2 transforms O_B to L_B.

21. L_BF is the median of triangle $A_1L_BC_1$ and by previous item F is the midpoint of L_BG. Thus F divide L_BM in ratio $2:1$. Then F is the centroid of triangle $A_1L_BC_1$.

22. Let point X be isogonally conjugated to L_B. Since L_B lies on the Feuerbach hyperbola (see part X), X lies on line OI. Prove that X lies on line A_1C_1. Let X'be the reflection of X in the bisector of angle B. Since OI passes through G', this reflection transforms OI to IG. Also it fixes line A_1C_1 and transforms line BX to line BL_B, i.e. to line $A'C'$. Lines $A'C'$ and A_1C_1 meet on IG as corresponding sidelines of triangles $A'B'C'$ and $A_1B_1C_1$. Thus X' lies on A_1C_1, and so X also lies on this line.

Part X

1. Take two vertices and the images of three arbitrary points on the line. They define some conic. Take an arbitrary fourth point on the line. When we conjugate the corresponding lines in two angles the conjugated lines intersect the conic at the same point because the reflection in the bisector saves the cross-ratios of four lines.

2. Clearly I is a common point. Suppose that there exists another common point X. Then its isogonally conjugated point X' also lies on the line and the hyperbola. But a line and a conic can't have three common points.

3. **Lemma.** A circumconic of a triangle is an equilateral hyperbola iff it passes through the orthocenter.

   Proof. The isogonal image of a circumconic is a line because five points define an unique conic. When the conic passes through the orthocenter the corresponding line passes through the circumcenter. This line meets the circumcircle at two opposite points. Thus the conic has two infinity points corresponding to two perpendicular directions.

   The problem immediately follows from the lemma.

   Let R the midpoint of arc BC of the circumcircle of ABC. Then OR is parallel to IB_1, and external angle $\angle BOR$ is twice greater than angle BRO, which is equal to $\angle B_1IR$. Thus the reflection of IB_1 in IR is the line parallel to OB. Therefore the tangent to the incircle in X is parallel to the tangent to the circumcircle in B, and this immediately yields the assertion of the problem.

4. Point I is isogonally conjugated to itself, A, B and C are isogonally conjugated to the common points of OI with the sidelines, H is isogonally conjugated to O. SinceG' is the homothety center of the incircle and the circumcircle, G' lies on OI. Thus Glies on the hyperbola. Similarly for N: N' is the second homothety center of the incircle and the circumcircle.

5. Let l meet the circumcircle in points X and Y. Then X and Y are isogonally conjugated to infinity points X' and Y' of the hyperbola, O is conjugated to H. Let Z be conjugated to the infinity point of l. Since $(X, Y, O, Z) = -1$, the cross-ratio of the conjugated points is the same. Project this cross-ratio from X' to line HZ'. This projection transforms Y' to the infinity point of this line. Since H and Z' are fixed, the projection of X' is the midpoint of HZ'. But the projection of X' lie on the tangent to hyperbola in X', i.e on the asymptote of the hyperbola. This yields that the midpoint of $Z'H$ is the center of hyperbola. Finally note that H is the homothety center of Euler circle and the circumcircle and Z' lies on the circumcircle as the image of infinity point Z'. Thus the center lies on the Euler circle.

6. See the solution in book “Geometrical properties of conics”.

7. Follows from previous item.

8. Since L_B is the orthocenter of triangle ACI, the soyght assertion follows from the lemma of problem 3.

9. Note that there exists exactly one point of line BI such that its polars wrt the hyperbola and the circle coincide. It is the common point of the diagonals of the quadrilateral formed by the common points of these two conics. Take now a common point of $BI$and FB_1. Its polars pass through the point which is the fourth harmonic for our point B and I. The polar wrt the circumcircle is perpendicular to BI because BI is a diameter. The polar wrt the hyperbola is also perpendicular to BI: the polar of F is the infinity line because F is the center of the hyperbola. The polar of B_1 is the line A_1C_1, because the corresponding points on lines AC and BG are harmonic. Thus the pole of $B_1F$is the infinity point of A_1C_1, i.e. the point corresponding to the direction perpendicular to BI. Therefore the common point of BI and FB_1 lies on tyhe common chord of the hyperbola and the circle.

Lemma. Let A and B be two point on an equilateral hyperbola. A circle with diameter AB meets the hyperbola at points P and Q. Then PQ passes through the center of the hyperbola .

Proof. Let H be the orthocenter of APQ. Since the hyperbola is equilateral H lies on it. Note that $HPBQ$ is a parallelogram. Since its vertices lie on the hyperbola its center is the pole of the infinity line, i.e. the center of the hyperbola.

Now we have that the common chord passes through the common point of BI and FB_1. Also it passes through the center F of the hyperbola. Therefore it coincide with FB_1.

10. Line GM is parallel to $A'C'$, and F is the midpoint of GL_B. Thus GM is the reflection of $A'C'$ in F. Line L_BB_0 is parallel to O_BF and BB_1. Thus L_BB_0 is the reflection of BB_1 in F. Since the Feuerbach hyperbola is symmetric wrt F, we obtain the assertion of the problem.

11. It is sufficient to prove that the poles of FB_0 and A_1C_1 lie on AC. The pole of A_1C_1 is point B_1 clearly lying on AC. The pole of FB_0 is the common point of polars of points F and B_0. But these both polars pass through the infinity point of AC.

12. Denote the reflection of B_1 in E as X. Then $B_1A_1XC_1$ is a parallelogram. Prove that the polar of A' is the line passing through A_1 and parallel to B_1C_1.

Lemma. The polar of A_1 wrt the Feuerbach hyperbola is line B_1C_1.

Proof. Consider passing through A_1 line BC. The fourth harmonic point for $A_1$and two common points of this line with the hyperbola (B and C) is the common point of B_1C_1 and BC. Take line AA_1. It meets the hyperbola at points A and G. The fourth harmonic point for these three points is the common point of B_1C_1 and AA_1. Thus the polar of A_1 is the line B_1C_1.

Use the lemma. Since A' lies on $B'C'$, its polar passes through the pole A_1 of this line. On the other hand A' lies on line FA_1. Thus its polar passes through the pole of this line. But this pole is the common point of the polars of F and A_1. The polar of $F$is the infinity line. By the lemma the polar of A_1 is line B_1C_1. Thus the pole of our line is the infinity point of line B_1C_1. Using the similar fact for point C' we obtain the sought assertion.

Part F.

1. Since triangle GC_1Q is isosceles and $QC_1Q''A$ is a parallelogram, we obtain that AGQ_1Q'' is an isosceles trapezoid. Similarly BGP_1P'' an isosceles trapezoid. By angles calculation we obtain that Q'', G and P'' are collinear. Then $P''Q'' = P''G + GQ'' = AC_1 + CA_1 = AB_1 + CB_1 = AC$.

2. The tangent in F is parallel to QP and $Q''P''$, also it is antiparallel to C_1A_1 wrt angle C_1FA_1. Thus quadrilateral is $Q''C_1A_1P''$ is cyclic. Since FG is the median of triangle FA_1C_1, it is the symedian of triangle $P''FQ''$. By angles calculation we obtain that FB and FG are isogonally conjugated wrt angle A_1FC_1. Thus FB is the median. FI and FG' are the radius of the circumcircle and the altitude of triangle FC_1A_1. Therefore they are the altitude and the line passing through the circumcenter of triangle $FQ''P''$.

3. Denote the common point of B_1T and BC as X. By Pascal theorem for points B_1, C_1, A_1, A_1, F, T line $A'X$ is parallel to A_1C_1. Let Y, Z, Z' be the common points of B_1C_1 and BC, B_1K and BC, $A'E$ and BC respectively. We have $(X, Z, A_1, Y) = (T, K, A_1, C_1) = -1 = (A'X, A'E, A'A_1, A'C_1) = (X, Z', A_1, Y)$. Thus $Z = Z'$.

4. Let GG' and BB' secondary intersect the Feuerbach hyperbola in points X and Y respectively. $(A, C, L_B, X) = (GA, GC, GL_B, GX) = (GC_1, GB_1, GBE, GG') = -1 = (AB, CB, A'C', BB') = (A, C, L_B, Y)$. Thus $X = Y$.

5. Prove that the homothety with center M and coefficient -2 transforms F to D. In fact this homothety transforms B_0 to B, FB_0 to BB' and fixes FG'. Then it transforms F to D. Since F on the Euler circle, D lies on the circumcircle.

6. The homothety from previous item transforms the midpoint of C_0A_0 to B_0, thus it transforms FO_B to L_BB_0. Then L_BB_0 passes through D.

7. In part X we proved that there exists an ellipse passing through A_0, A_1, B_0, B_1, C_0, C_1 and F. Prove that our three lines pass through its center. Line FO_B passes through A_0C_0 and B_0C_1, thus it is a diameter conjugated to direction AC. This yields also that the tangents to ellipse in its common points F and S with FO_B are parallel to AC. By Pascal theorem for points B_0, C_0, C_1, A_1, A_0, B_0 the tangent in B_0 is parallel to C_1A_1. Thus B_0E is a diameter.

   Solutions of other items may be on the official site of conference.