TROPICAL GEOMETRY

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A brief overview. [6]

Hilbert's 16th problem asks *what could be the number and mutual arrangement of curves which form the subset of the plane given by an equation* $\sum\limits_{i+j\le d} a_{ij}x^iy^j = 0$. A more accurate statement and examples are given in part A. [7] The aim of this series of problems is to outline an approach to the "existence" part of Hilbert's 16th problem for $d = 6$ (see Main Theorem below).

It is not easy to determine the number and mutual arrangement of curves for the subset of the plane given by a polynomial in two variables with certain concrete coefficients (even using a modern computer). While solving the problems of part B you will find the statement of main lemma which allows to do it for polynomials of certain specific type. You will see how *tropical geometry* appears naturally while drawing of subsets given by the equations of type $\sum\limits_{i+j\le d}(a_{ij}x^iy^j)^N = 0$, where N is a large odd number. Using tropical geometry you will be able to construct such subsets with distinct mutual arrangement of ovals.

The basic ideas of tropical geometry are elementary. Replace multiplication by addition, and addition by an operation related to addition via the same *distributive* law, like multiplication is related to addition. As such an operation one can take *maximum* $\max\{a, b\}$ of the pair of numbers a and b. Under this transformation the function $\sum\limits_{i+j\le d} b_{ij}x^iy^j = 0$ transforms to the function (check it!): $f(x, y) = \max\limits_{i+j\le d}(ix + jy + b_{ij})$. The set of "break points" of the function is called *the tropical curve*.

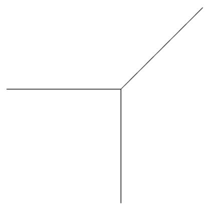

Figure 1.

For instance, a line in the plane is given by the equation $Ax+By+C = 0$. Left part of this equation turns to the function $f(x, y) = \max\{x + a, y + b, c\}$ under our transformation. the set of "break" points of the function $f(x, y)$ looks like shown in figure 1 (check it!). This way *the tropical line* is defined. Tropical lines have many properties of Euclidean lines. Part C of the project deals with "experimental" investigation of these properties.

A. Examples of algebraic curves.

A *polynomial* (in two variables) is a function $F : \mathbb{R}^2 \to \mathbb{R}$ for which there exist numbers d and a_{ij}, $0 \le i, j \le d$, such that $F(x, y) = \sum\limits_{i+j\le d} a_{ij}x^iy^j$. You can use without proof the following non-trivial fact: *for given function F such numbers are unique up to increasing d and taking all the new a_{ij} to be zeroes.*

The *zero set* of the polynomial F is $F^{-1}(0) := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$.

A1. Is F uniquely determined by $F^{-1}(0)$?

A2. Which of the following sets are zero sets of polynomials?

(a) a line; (b) a circle; (c) a point; (d) a segment; (e) the union of 2 lines;

(f) the "pig" (union of 6 circles) in figure 2.

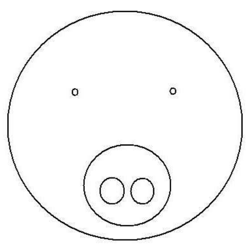

Figure 2.

The *degree* of a polynomial is the least possible d for which there exist the required a_{ij}. (The degree is the maximal d such that $a_{i,d-i} \ne 0$ for some representation of the polynomial and for some number i.)

A3. (a) How many points there could be in the intersection of the zero set of a polynomial of degree d and a line?

(b) The zero set of a polynomial of odd degree is unbounded (i.e. is not contained in a disk).

[6] Do not worry if you do not understand something in this brief overview. You can omit it and start solving problems from either part A or part C.

[7] For $d \le 5$ the answer was known as early as in 19th century. Hilbert stated his problem for $d = 6$. The solution for this case was obtained by Gudkov. For $d = 7$ the problem was solved by Viro using methods of tropical geometry. For $d \ge 8$ the problem is open.

A polynomial F is *reducible*, if $F = G \cdot H$ for some polynomials G and H.

Curves[8]**.** A function $\gamma : [a, b] \to \mathbb{R}$ is *differentiable* at the point t_0, if for some number A and any $\varepsilon > 0$ there exist δ such that for any

$$t \in (t_0 - \delta, t_0 + \delta) \quad \text{we have} \quad |\gamma(t) - \gamma(t_0) - A(t - t_0)| < \varepsilon |t - t_0|.$$

A map $\gamma : [a, b] \to \mathbb{R}^2$ can be considered as an ordered pair of functions $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$. A map $\gamma : [a, b] \to \mathbb{R}^2$ is *differentiable* if both functions γ_1, γ_2 are differentiable.

A *(smooth) curve* in the plane is a differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^2$ (or $\gamma : [a, b] \to \mathbb{R}^2$).

In problems A4.cfg and A7 you only need to give an example of a polynomial; proving its properties is not required.

A4. (a) There is an irreducible polynomial of degree 3 whose zero set contains a closed curve.

(b) The same for degree 4.

(c) There is an irreducible polynomial of degree 4 whose zero set consists of two closed curves one inside the other.

(d) If the zero set of a polynomial of degree 4 contains two closed curves one inside the other, then the zero set contains no other points.

(e) Is the analogue of (d) correct for an irreducible polynomial of degree 5?

(f) There is a polynomial of degree 4 whose zero set contains 4 closed curves.

(g) There is a polynomial of degree 4 whose zero set contains 3 closed curves.

Ovals . Connected components of the zero set of a polynomial are called *branches.* (Existence of unbounded branches makes the investigation of zero sets harder.) For an unbounded branch B the lines joining the origin O with the points of B have a "limit" line. Two unbounded branches are *elementary equivalent* if their "limit" lines coincide.

A5. The infinite branches of hyperbola $xy = 1$ are elementary equivalent.

Two infinite branches are *equivalent* if there is a sequence of branches joining them, in which sequence each two consecutive branches are elementary equivalent. A zero set is *nondegenerate* if it is a disjoint union of smooth curves. An *oval* of a non-degenerate zero set of a polynomial is either a closed curve (contained in the zero set) or an equivalence class of unbounded branches. (Note that this definition is different from the "correct" one given in textbooks.)

A6. Find all h such that the zero set is non-degenerate and find the number of ovals for the polynomial
(a) $xy(x + y - 1) - h$. (b) $x^3 - x + h - y^2$. (the answer could depend on h).

A7. There is a polynomial of degree 5 whose zero set is non-degenerate and consists of 7 ovals.

Hilbert's 16th problem. *What could be the number and mutual arrangement of ovals of a non-degenerate zero set of a polynomial of degree d?*

We do not assign any formal meaning to the words 'mutual arrangement'. Such a meaning can be assigned, but requires *projectivization* of a polynomial.

Main Theorem. (a) *There is a polynomial of degree* 6 *whose zero set is non-degenerate and consists of* 11 *ovals.*

(b) *There are three polynomials of degree* 6 *each whose zero sets are non-degenerate and consist of* 11 *ovals each, with different mutual arrangement of ovals.*

B. Tropical curve as a limit of algebraic curves.

B1. Draw the zero sets of
(a) $x - y - 1$; (a') $x^{1001} - y^{1001} - 1$;
(b) $x + y - 1$; (b') $x^{1001} + y^{1001} - 1$;
(c) $xy = x + y$; (c') $x^{1001}y^{1001} = x^{1001} + y^{1001}$;
(d) $x^2 + y^2 - 4x - 4y - 2 = 0$; (d') $x^{2002} + y^{2002} - 4^{1001}x^{1001} - 4^{1001}y^{1001} - 2^{1001}$;
(e') $x^{3003} + 2^{1001}x^{1001}y^{2002} - 3^{1001}x^{1001}y^{1001} + y^{2002} - x^{1001} - 2^{1001}$.

Denote by

$$F_N(x, y) = \sum_{i+j \le d} (a_{ij}x^i y^j)^N$$

[8] These definitions are required only for the accurate proofs of problem A4.

a family of polynomials depending on an *odd* number $N \geq 1$. Under the transformation of variables $u = x^N, v = y^N$ each polynomial F_N goes to the polynomial $\sum_{i+j\leq d} a_{ij}^N u^i v^j$ of degree d. So for solution of the Hilbert 16th problem it is worth to determine the number and mutual arrangement of ovals of $F_N^{-1}(0)$.

B2. The number of ovals of $F_N^{-1}(0)$ can be different from that of $F_1^{-1}(0)$.

Denote by B_R the ball of radius R centered at 0.

B3. (a) For each $\varepsilon, R > 0$ there is $N_0 > 0$ such that for each odd $N > N_0$ the intersection of the zero set of $x^{2N} - x^N - y^N$ with B_R is contained in ε-neighborhood of the union of the lines $x = 0$, $x = 1$, $x = y$ and the parabola $y = x^2$.

(b) For each $\varepsilon, R > 0$ there is $N_0 > 0$ such that for each odd $N > N_0$ the set $F_N^{-1}(0) \cap B_R$ is contained in ε-neighborhood of the union of the zero sets of all the polynomials $a_{ij}x^iy^j - a_{kl}x^ky^l$, in which $(i,j) \neq (k,l)$, $i + j \leq d$, $k + l \leq d$.

Denote by $\mathbb{R}_+ := [0, +\infty)$ the set of positive numbers and by $\mathbb{R}^2_+ := [0, +\infty)^2$ the angle defined by the inequalities $x > 0$, $y > 0$. Define a map $LOG : \mathbb{R}^2_+ \to \mathbb{R}^2$ by $LOG(x, y) = (\log_2 x, \log_2 y)$.

B4. (abcde) Draw the LOG-image of the intersection with $\mathbb{R}^2_+$ of the zero sets of polynomials (a'b'c'd'e') of B1.

A *tropical polynomial* is a function

$$f(x, y) := \max_{i+j\leq d}(ix + jy + b_{ij}).$$

$$\text{Let} \quad f^{pq} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = px + qy + b_{pq}\}.$$

The union of intersections of different f^{pq} is *a tropical curve*. (This is the set of "break points" of f.)

Assume further that all $a_{ij} \neq 0$ for $i + j \leq d$. The tropical curve *corresponds* to the family of polynomials F_N, if $b_{ij} = \log_2 |a_{ij}|$. This definition is motivated by important problem B6b below.

B5. Draw the tropical curve corresponding to the family of polynomials
(a) $(ax)^N + (by)^N + c^N$? (b) $(ax^2)^N + (2bxy)^N + (cy^2)^N$? (the answer could depend on a, b, c.)

Denote by Δ_R the triangle given by the inequalities $x \geq -R, y \geq -R, x + y \leq R$.

B6. (a) For each $\varepsilon, R > 0$ there is N_0 such that for $N > N_0$ the intersection of the LOG-image of the zero set of the polynomial $x^{2N} - x^N - y^N$ with the triangle Δ_R is contained in ε-neighborhood of the union of the ray $y = 2x, x \geq 0$ and the ray $x = 0, y \geq 0$.

(b) For each numbers $\{a_{ij}\}_{i+j\leq d}$ and $\varepsilon, R > 0$ there is $N_0 > 0$ such that for $N > N_0$ the set $LOG(F_N^{-1}(0) \cap \mathbb{R}^2_+) \cap \Delta_R$ is contained in ε-neighborhood of the intersection of the tropical curve corresponding to F_N.

C. Tropical lines and circles.

This part of the project is an contest in art: it is suggested to check the theorems of tropical geometry experimentally by drawing accurate figures. Selected figures will be exposed for public viewing. "Problems" C1-C10 are not graded (although for accurate statement and proving some of these assertions additional points would be awarded). Ignore part of a "problem" if you do not know the corresponding theorem of Euclidean geometry. The whole part C of the project is not required for the solution of the Hilbert 16th problem and can be skipped.

Consider the plane with fixed Cartesian coordinate system. A *tropical line* (*"leg"*) is a union of three rays with common origin (called the *vertex*), one of them going "west", the other going "south" and the third going "north-east".

C1. There are different tropical lines intersecting at two different points.

Two points are *in general position* if the Euclidean line passing through these points is not parallel either to coordinate axes or to the line $x = y$.

C2. (a) For each two points in general position there is a unique tropical line passing through these points.
(b) If the vertices of two tropical lines are in general position, then the lines have the only common point.

Two tropical lines are *parallel* if the vertex of one lies on the "north-eastern" ray of the other.

C3. If a point A is in general position with the vertex of a tropical line b, then there is a unique tropical line passing through A and parallel to b.

Two tropical lines are *perpendicular* if the Euclidean lines containing their "north-eastern" rays are symmetric with respect to the line $x = y$.

C4. If a point A is in general position with the vertex of a tropical line b, then there is a unique tropical line passing through A and perpendicular to b.

A *tropical triangle* (*"spider"*) is the union of three tropical lines whose vertices are (pairwise) in general position.

C5. Draw figures to tropical analogues of the following theorems.
(a) The heights of a triangle intersect at a common point.
(b) the Pappus theorem.
(c) The Desargue theorem.
(d) The Sondat theorem.

For given points A and B a *tropical circle* (*"heron"*) is the set of points X for which there are orthogonal tropical lines, one of them passing through A and X and the other passing through B and X. (Recall that there could be different tropical lines passing through A and X.)

C6. (a) Draw a tropical circle. How does this set depend on A, B?
(b) Is it true that each tropical triangle has a circumscribed tropical circle?
(c) The Pascal theorem.

C7.* Define the *tropical middle point of a tropical segment* so that the tropical medians of a tropical triangle would intersect in a common point.

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The main problem's complect consists of two parts: the end of Part B and a new part D. Problems of part D use (except some explicitly indicated cases) neither notions nor results of previous parts of the project. So one may solve these problems without taking part in previous parts.

B. The Viro patchworking theorem.

B3. For each $\varepsilon, R > 0$ there is $N_0 > 0$ such that for each odd $N > N_0$ the intersection of the zero set of the polynomial

(c) $x^{2N} - x^N - y^N$ with the disk B_R and with the first coordinate quarter ($x > 0$, $y > 0$) is contained in the ε-neighborhood of the union of the sets

$$\{(1, y) \mid 0 \le y \le 1\}, \quad 0 \le x = y \le 1 \quad \text{and} \quad y = x^2 \ge 1.$$

(d) $x^{2N} - x^N - y^N$ with the disk B_R and with the second coordinate quarter ($x < 0$, $y > 0$) is contained in the ε-neighborhood of the union of the set that is symmetric to the union from (c) w.r.t. Oy.

B7. State and prove the analogues of B3d for the third and the fourth coordinate quarters.

B8. The intersection of the zero set of the polynomial $x^{2N} - x^N - y^N$ with the third coordinate quarter is empty.

B9. For each $\varepsilon, R > 0$ there is $N_0 > 0$ such that for each odd $N > N_0$ the intersection of the zero set of the polynomial $x^{2N} - x^N - y^N$ with the disk B_R and with

(a) the first coordinate quarter is contained in the ε-neighborhood of the union of the sets $\{(1, y) \mid 0 \le y \le 1\}$ and $y = x^2 \ge 1$.

(b) the second coordinate quarter is contained in the ε-neighborhood of the union of the sets $0 \le -x = y \le 1$ and $y = x^2 \ge 1$.

B10. State and prove the analogue of B9 for the fourth coordinate quarter.

Let us state the Viro patchworking theorem that allows to find the number and mutual arrangement of ovals for certain special algebraic curves.

B11. Each tropical curve is a finite union of segments and rays.

Definition of the Viro curve and its ovals. Take the tropical curve corresponding to $\{a_{ij}\}$. The tropical curve is a finite union of *edges* (segments and rays) that intersect at *vertices* (i.e. at common points of edges). A *face* of the tropical curve is a connected component if its complement in the plane. To each face there corresponds a pair (p, q) of integers such that $px + qy + \log_2 |a_{pq}| = \max\limits_{i+j \le d,\ a_{ij} \ne 0} (ix + jy + \log_2 |a_{ij}|)$ for points (x, y) of this face, and the sign of $a_{p,q}$. *In this definition we use not $\{a_{ij}\}$ but the tropical curve whose faces are marked with pairs of integers and signs.*

Make a parallel transfer so that the vertices of the tropical curve would move into the angle $x > 0, y > 0$. Define $U_{p,q,00}$ to be the image of the face of the tropical curve marked by (p, q) under this parallel transfer. Let $U_{p,q,01}, U_{p,q,10}$ and $U_{p,q,11}$ be the symmetric images of $U_{p,q,00}$ under the symmetries with respect to the x-axis, y-axis and $(0, 0)$, respectively. Extend the given disposition of signs from the first coordinate quarter to the whole plane as follows: under the symmetry of U_{pq} w.r.t. the x-axis the sign is multiplied by $(-1)^q$, while under the symmetry of U_{ij} w.r.t. the y-axis the sign is multiplied by $(-1)^p$. (Thus $\operatorname{sgn} U_{pq,st} = (-1)^{ps+qt} \operatorname{sgn} U_{pq,00}$.) Define *the Viro curve* to be the union $\cup\{U_\alpha \cap U_\beta \mid \operatorname{sgn} U_\alpha \ne \operatorname{sgn} U_\beta\}$ of those edges of the tropical curve that split faces of different signs (see Figure). Two unbounded connected components of the Viro curve are

- *elementary equivalent* if they contain rays symmetric w.r.t the origin $(0, 0)$.
- *equivalent* if there is a sequence of components joining them, in which sequence each two consecutive components are elementary equivalent.

An *oval* of the Viro curve is either a closed broken line contained in the Viro curve or an equivalence class of unbounded connected components.

You can use the following theorem without proof:

The Harnak Theorem. *A non-degenerate zero set of a polynomial of degree d cannot have more than* $\frac{(d-1)(d-2)}{2}+1$ *ovals.*

B12.* The Viro patchworking theorem. *Let the Viro curve assigned to the family of polynomials* $F_M = \sum_{i+j\le d}(a_{ij}x^iy^j)^M$ *with all* $a_{ij}\neq 0$ *contain exactly* $\frac{(d-1)(d-2)}{2}+1$ *ovals. Then there exist N such that the zero set of the polynomial* $\sum_{i+j\le d} a_{ij}^N u^i v^j$ *is non-degenerate, and the number and mutual arrangement of the ovals are the same as those of the corresponding Viro curve.*

D. Construction of examples in the Hilbert 16th problem.

The aim of part D is to describe tropical curves using purely combinatorial method and to obtain a purely combinatorial construction of examples in 16th Hilbert problem.

Let us recall that *tropical curve of degree d* is the set of "break points" of graph of the function $\max_{i+j\le d}\{ix+jy+b_{ij}\}$ (see the details above, after problem B4).

D1. (a) Check that a tropical curve of degree 1 looks like picture 1. (Compare with the definition of a tropical line in part C).

(b) Each vertex of a tropical curve is contained in at least 3 edges.

To any edge of a tropical curve assign its *multiplicity* as follows. Suppose that value $ix+jy+b_{ij}$ is maximal in one of faces bounded by this edge, and value $i'x+j'y+b_{i'j'}$ is maximal in the other one. So the line, which contains the given segment, has the equation $(i-i')x+(j-j')y+(b_{ij}-b_{i'j'})=0$. We define *multiplicity* of the given edge as the greatest common divisor of numbers $i-i'$ and $j-j'$.

In pictures we shall denote the multiplicate edges of a tropical curve with double (triple, and so on) lines.

D2. Tropical curves of degree d have the following properties:

(a) The slope of any edge is a rational number.

(b) For any vertex the following balance condition holds. Denote by v_i a vector beginning at the given vertex parallel to i-th edge starting from the vertex, and equal to the shortest vector with integer coordinates and given direction, multiplied by edge's multiplicity. Then $\sum v_i = 0$.

(c) There are $3d$ infinite edges (counted with multiplicity), d of them are directed (strictly) to the "west", d — to the "south" and d — to the "north-east" with slope angle 45°.

D3. (a) One may uniquely restorate a tropical polynomial $\max_{i+j\le d}\{ix+jy+b_{ij}\}$ (up to adding a constant) by its tropical curve.

(b) If the edges of a graph in the plane are segments and rays with given multiplicities, and the conditions (a), (b), (c) of problem D2 are satisfied, then the graph is a tropical curve of degree d.

If two tropical curves have the same combinatorial type of their graphs and the same slopes of their edges (but not nesessary their lengths and positions), we shall say that these curves have it the same configuration.

D4. Draw 5 different configurations of tropical curves of degree two.

All information required for solving the following problems you can find in the paragraph "Definition of Viro curve and its ovals" contained in the previous part.

D5. What maximal number of ovals may have Viro curve if $d=$ (a) 2; (b) 3; (c) 4; (d) 5? (We do not require the proof of maximality. Compare your answer with problems A4f and A7).

D6*. Write down a computer program which:

(a) draws all configurations of tropical curves of given degree d;

(b) given a tropical curve configuration and given the set of signs "plus, minus" assigned to all the faces U_{ij} of its complement — the program checks the number of Viro curve ovals.

D7*. (ab) Prove the Main Theorem (you may use Viro patchworking theorem without proof).

SOLUTIONS

A1. Answer:no. For example, the line $x = 0$ is a set of zeros for different polynomials $F(x, y) = x$ and $G(x, y) = x^2$.

A2. Answer: a, b, c, e, f.

Examples. (a) Any line on the plane has an equation $Ax + By + C = 0$ for some numbers A, B, C.

(b) Equation of a circle: $(x - x_0)^2 + (y - y_0)^2 - R^2 = 0$, where (x_0, y_0) are coordinates of centre, R is radius.

(c) Equation of a point (x_0, y_0): $(x - x_0)^2 + (y - y_0)^2 = 0$.

(e) Equation of a unite of two lines: $(Ax + By + C)(ax + by + c) = 0$, where $Ax + By + C = 0$ is an equation of the first line, $ax + by + c = 0$ — of a second one.

(f) Equation of a unite of 6 circles: $\prod\limits_{k=0}^{6} \left((x - x_k)^2 + (y - y_k)^2 - R^k\right) = 0$, where $(x-x_k)^2+(y-y_k)^2-R_k^2 = 0$ is an equation of k-th circle.

Impossibility in point (d) is consequence of Problem A3a.

A3. (a) Let us parametrize the line l: $x = x_0 + \alpha \cdot t$, $y = y_0 + \beta \cdot t$. Substituting these formulas in the polynomial, we'll get a new polynomial $P(t)$, its degree no more than d. So polynomial $P(t)$ has no more than d real roots, or equals to zero everywhere. Now let us prove that for any $d' < d$, there exist a curve of degree d and a line l such one, that they have d' points of intersection. Consider d lines, which are differ from l, and such that exactly $d - d'$ of them are parallel to l. The product of their equations is the polynomial we need.

(b) Let d be the degree of given polynomial $F(x, y) = \sum\limits_{i+j\le d} a_{ij}x^iy^j$. We'll show that there exist some non-degenerate change of coordinates $x = \alpha_1x' + \beta_1y'$, $y = \alpha_2x' + \beta_2y'$ (the word "non-degenerate"means that $\alpha_1\beta_2 - \alpha_2\beta_1 \ne 0$), such that after it the monomial $(x')^d$ will have non-zero coefficient.

Coefficient $A(\alpha_1, \alpha_2)$ of monomial $(x')^d$ equals $\sum\limits_{i+j\le d} a_{ij}\alpha_1^i\alpha_2^j$. Numbers a_{ij} aren't equal zero (at least, some of them), so, there exist such α_1 and α_2, that at least one of them isn't equal 0, and $A(\alpha_1, \alpha_2) \ne 0$. Now we take coefficients β_1 and β_2 not proportional to α_1 and α_2 (i.e., $\alpha_1\beta_2 - \alpha_2\beta_1 \ne 0$), and it will be the change we seek for.

Now let us return to solving our problem. The change from the Lemma transforms bounded sets to bounded ones, so we may suppose that monomial x^d has non-zero coefficient. As d is odd number, so for any y the equation $F(x, y) = 0$ has some solution. So $F^{-1}(0)$ is unlimited.

A4. (a) For instance, take the polynomial $f = xy(x + y - 1) + \frac{1}{100}$.

Denote by ϕ the zero set of this polynomial. Let us prove that this polynomial is irreducible. Indeed, otherwise there are polynomials g and h, such that $f = gh$. Then one of them is a polynomial of degree 1 and so ϕ contains a line. This line must have a common point with one of the lines Ox and Oy. But it can't be true because ϕ is disjoint with Ox and Oy. Thus f is irreducible.

Coordinates x of the intersection of line $y = c$ with ϕ satisfy the equation $x^2 + (c - 1)x + \frac{1}{100c} = 0$. The discriminant $D = D(c)$ of this equation is equal to $(c - 1)^2 - \frac{1}{25c}$. The equation $D(c) = 0$ is equivalent to the equation $f(c) := 25c(c - 1)^2 - 1 = 0$. This equation has degree 3 and thus has no more than 3 roots. Since $f(\frac{1}{100}) < 0$, $f(\frac{1}{2}) > 0$, $f(1) < 0$, $f(2) > 0$, it follows that two roots c_1 and c_2 of the equation $f(c) = 0$ belong to the interval $(0, 1)$, and the third root belongs to the interval $(1, 2)$. Therefore $D(c) = 0$ precisely in two points c_1 and c_2 of the interval $(0, 1)$, and $D(c) > 0$ for any $c \in (c_1, c_2)$ and $D(c) < 0$ for remaining points of the interval $(0, 1)$. (We assume w. l. g. that $c_1 < c_2$.) Thus for c equal either c_1 or c_2 the straight line $y = c$ intersects ϕ exactly at one point. Therefore for $c \in (c_1, c_2)$ the straight line $y = c$ intersects ϕ at two points $(x_1(c), c)$ and $(x_2(c), c)$, where $x_{1,2}(c) = \dfrac{\pm\sqrt{D} - (c - 1)}{2}$. For remaining values $c \in (0, 1)$ the straight line $y = c$ does not intersect ϕ.

Define the curve

$$\gamma : [c_1, 2c_2 - c_1] \to \mathbb{R}^2 \quad \text{by the formula} \quad \begin{cases} (x_1(t), t) & t \in [c_1, c_2] \\ (x_2(2c_2 - t), 2c_2 - t) & t \in [c_2, 2c_2 - c_1] \end{cases}$$

Since the functions $x_1(c)$ and $x_2(c)$ are differentiable, it follows that the map $\gamma(t)$ is differentiable at all points except c_2. Since $2c_2 - t = t$ for $t = c_2$ and $(x_1)'(c_2) = (x_2)'(c_2)$, the map $\gamma(t)$ is smooth at all points. Now it is clear that $\gamma(I)$ is a closed curve contained in ϕ.

(b) Hint. Consider the polynomial $(x + 1)(x - 1)(y + 1)(y - 1) + \frac{1}{100}$.

(c) Hint. Consider the polynomial $(x^2 + y^2 - 1)(x^2 + y^2 - 9) + \frac{1}{100}$.

(d) Suppose the contrary: there exist at least one other point X. Consider some point Y inside the inner closed curve. Then the line XY intersects the set of zeros of the given polynomial in 5 or more points. It contradicts the statement of Problem A3(a).

(e) Answer: no. Hint. Consider the polynomial $x(x^2+y^2-1)(x^2+y^2-9)+\frac{1}{100}$.

(f) Hint. Consider the polynomial $(x^2+2y^2-3)(2x^2+y^2-3)+\frac{1}{100}$.

(g) Hint. Consider the polynomial $(x^2+y^2-1)(x-y-1)(x+y-1)+\frac{1}{100}$.

A5. Direction of a line OM, on which lie points 0 (beginning of coordinates) and $M(x,y)$ on hyperbola branch in the first quadrante, tends to direction of the line Ox (axis) when $x\to+\infty$. So Ox is "limit line"for hyperbola $xy=1$ branch in the 1st quadrante. Similarly, this line is a "limit line"for the other branch of hyperbola. So hyperbola's branches are equivalent.

Definition. Two unbounded branches are *elementary equivalent*, if they have a common "limit line".

A6. (a) Answer: one oval, if $h<0$; two ovals, if $h\in(0,1/27)$, one oval, if $h>1/27$. If $h=0$ or $h=1/27$, the algebraic curve is degenerate. Here is the proof.

Denote $f(x,y):=xy(x+y-1)+h$. Further, denote the points of intersection of lines Ox, Oy and $x+y-1=0$ and regions, into which the plane is divided by the lines as follows:

$$A:=(1,0),\quad B:=(0,1),$$

$C:=(0,0)$, $X:=\{(x,y)\mid x>0,y>0,x+y<1\}$, $X_A:=y<0,x+y>1$, $X_B:=x<0,x+y>1$, $X_C:=x<0,y<0$, $Y_A:=x<0,y>0,x+y<1$, $Y_B:=x>0,y<0,x+y<1$, $Y_C:=x>0,y>0,x+y>1$.

Obviously $f(x,y)=h$ if (x,y) belongs to one of lines Ox, Oy or $x+y-1=0$, and $f(x,y)<h$ when (x,y) belongs to one of regions X_A, X_B, X_C or X, and $f(x,y)>h$ when (x,y) belongs to one of regions Y_A, Y_B и Y_C. So if $h>0$ then zeros of polynomial $f(x,y)$ may lie only in X_A, X_B, X_C and X, and if $h<0$ they may lie only in Y_A, Y_B and Y_C.

Suppose $h<0$. Denote $y_A:=Y_A\cap f^{-1}(0)$. Definitions of y_B and of y_C are similar.

Let's prove that y_A is a connected componenta of the set $f^{-1}(0)$ of zeros of f. Coordinates x of points of intersection lines $y=c$ and $f^{-1}(0)$ are roots of the equation $x^2+(c-1)x+\frac{h}{c}=0$. The discriminant $D=D(c)$ of this equation equals to $(c-1)^2-\frac{4h}{c}$. As $h<0$ so for any $c\in R_+$, $D(c)>0$. It follows, that any line $y=c$, where $c\in R_+$, intersects F in two points (no more no less) namely $(x_{1,2}(c),c)$ such that $x_{1,2}(c)=\dfrac{\pm\sqrt{D}-(c-1)}{2}$. Denote

$$\gamma: R_+\to\mathbb{R}^2 \quad\text{by formula}\quad \Big\{(x_2(t),t)\quad t\in R_+\,.$$

The function $x_2(c)$ is smooth, so the curve $\gamma(t)$ is smooth also. $\gamma(R_+)=y_A$ implies that y_A is a connected componenta of the set $f^{-1}(0)$ of zeros of f. Similarly y_B и y_C are connected componentas of the same set. It is easy to prove that the direction of line Ox is a "limit direction" for the branch y_C. Similarly, this direction is a "limit direction" for branch y_A. So branches y_A and y_C are elementary equivalent. Similarly, branches y_A and y_B are elementary equivalent, as for these branches the direction of line $x+y-1=0$ is a "limit one". So, branches y_A, y_B and y_C are equivalent and form one oval.

If $h=0$, then an algebraic curve f is a degenerate one. Suppose $h>0$. Denote

$$x:=X\cap f^{-1}(0),\quad x_A:=X_A\cap f^{-1}(0),\quad x_B:=X_B\cap f^{-1}(0)\quad\text{and}\quad x_C:=X_C\cap f^{-1}(0).$$

One proves that x_A, x_B and x_C are connected and equivalent in the same way as in the case $h<0$. So they form one oval for any $h>0$. If the set x is non-empty and has more than one point, it is an oval (the proof is similar to solution of Problem A4(a)).

Let us prove that the set of points x isn't empty only if $h\in(0,\frac{1}{27}]$. It is clear that x is empty if and only if $D(c)<0$ for any $c\in(0,1)$. Derivative $D'(c)>0$ if $c\in(0,1/3)$, $D'(c)=0$ if $c=1/3$, $D'(c)<0$ if $c\in(1/3,1)$. So the function $D(c)$ has its maximum on intervale $(0,1)$ in the point $c=1/3$. So $D(c)<0$ for any $c\in(0,1)$ if and only if $D(1/3)=4/9-\frac{4h}{c}<0$, i.e. $h>1/27$. If $h=1/27$, then the set x consists from only one point. So, if $h\in(0,1/27)$, then the set of zeros $f^{-1}(0)$ consists from two ovals, if $h=1/27$, then the algebraic curve is degenerate, if $h>1/27$ then the set $f^{-1}(0)$ has only one oval.

(b) Answer: one oval if $h\in(-\frac{2}{3\sqrt{3}},\frac{2}{3\sqrt{3}})$, two ovals if $h\in(-\infty,-\frac{2}{3\sqrt{3}})$ and $h\in(\frac{2}{3\sqrt{3}},\infty)$, an algebraic curve $x^3-x+h-y^2$ is degenerate if $h=\pm\frac{2}{3\sqrt{3}}$. Hint. The situation is similar to (a).

A7. Hint. Consider the polynomial $x((x-1)^2+y^2-2)((x+1)^2+y^2-2)+\frac{1}{100}$.

B1. (a') *Hint.* See figure 3.a'.Why the picture is right one? It is clear that the line $x+y=0$ is an axis of symmetry for the set of zeros of our polynomial. So we may study only the case $y>-x$. Moreover. it lies under the line $y = x$. The set of zeros intersects with coordinate axes in points $(0, -1)$ and $(1, 0)$. If $x = 1 + \epsilon$ ($\epsilon > 0$) holds $(1001\epsilon)^{\frac{1}{1001}} < y < 1 + \epsilon$. So, if ϵ is sufficiently small, y may take values from 0 to 1. If $\epsilon > 1/1001$, then x is approximately equal to y. If $1 - \epsilon < x < 1$ (ϵ is sufficiently small) y may take values from -1 to 0. The case $y < x$ is analogous.

(b') *Hint.* See figure 3.b'.Everything is analogous to (a').

(c') *Hint.* See figure 3.c'.

(d') *Hint.* See figure 3.d'.

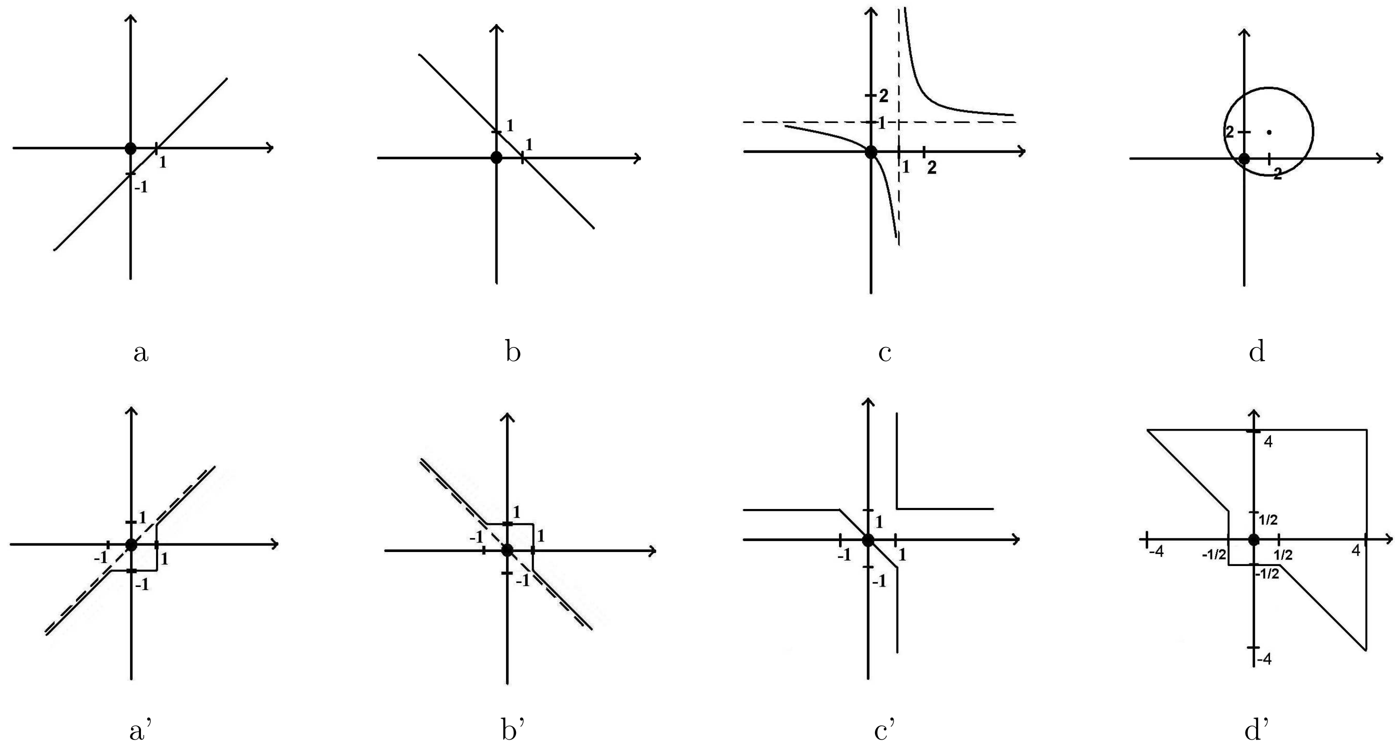


Figure 3.

B2. Let $F(x, y) = x^3 - px + q - y^2$, where $p, q > 0$. Then the set of zeros of the polynomial F consists of two ovals, if a polynomial $f(x) = x^3 - px + q$ has three real roots, and it consists of one oval, if $f(x)$ has one real root. After solving the equation $f'(x) = 0$, we'll see, that $f(x)$ has a local maximum in the point $x_1 = -\sqrt{p/3}$ and a local minimum in point $x_2 = \sqrt{p/3}$. It follows, that $f(x)$ has three roots if and only if $f(x_2) < q < f(x_1)$, i.e., $4p^3 > 27q^2$. Analogously, we can prove that a set of zeros of a polynomial $F_N(x, y)$ consists of two ovals if $4p^{3N} > 27q^{2N}$ and of one oval otherwise. It is obvious that if $1 < \frac{p^3}{q^2} < \frac{27}{4}$ the first inequality doesn't hold, and the second holds for sufficiently big N.

B3. (a) Point (a) is a specific case of (b).

(b) *Hint.* Suppose that there is a point (x, y) such that in it values of all monomials $a_{ij}x^iy^j$ differ by their modules, and $|a_{kl}x^ky^l| > |a_{ij}x^iy^j|$ for all pairs $(i, j) \neq (k, l)$. Then, when $N \to \infty$, we have $\left|\frac{a_{ij}x^iy^j}{a_{kl}x^ky^l}\right|^N \to 0$. So for any sufficiently big N $|a_{ij}x^iy^j|^N$ is more than the sum of modules of all the rest monomials, so the equality $F_N(x, y) = 0$ is impossible. So, for sufficiently big N the set $F_N^{-1}(0)$ tends to some subset of the union of sets, which are defined by equalities of the type $|a_{ij}x^iy^j| = |a_{kl}x^ky^l|$.

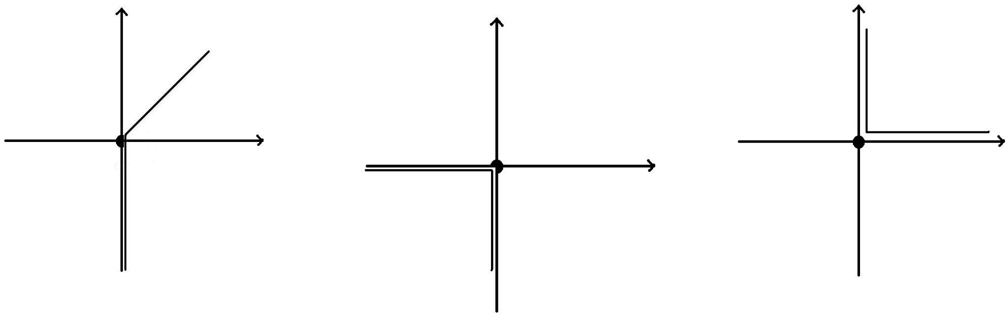
(c) It follows from the statement of previous problem. One must consider it for a polynomial $F_N = x^{2N} - x^N - y^N$.

(d) A set of zeros of a polynomial $x^{2N} - x^N - y^N$, which lie in a second quadrant, is symmetric with respect to ordinate axis to the set of zeros of a polynomial $x^{2N} + x^N - y^N$, which lie in a first quadrant. So our statement is a consequence of the statement of point (b).

B4. (a) *Hint.* See figure 4.a. Let us explain why the picture is correct. It is clear that the set of zeros of the function $f(x, y) := 2^{1001x} - 2^{1001y} - 1$ lies on the right of the axis Oy. If $y < 0$, then x is approximately equals to zero, and If $y > 0$, then x is approximately equals to y.

(b) *Hint.* See figure 4.b. Everything is analogous to (a).

(c) *Hint.* See figure 4.c.



a b c d

Figure 4.

B6. (a) We may follow the method used in problem B3b, and we'll see that the intersection of zero set with the 1st coordinate quadrant lies near the union of sets $x^2 = y \geq x$,$x^2 = x \geq y$, $x = y \geq x^2$. But given polynomial has the same signs of coefficients of monomials x^N and y^N, so F_N cannot equal zero near the last set. Logarithmical map brings the first named set to the ray $y = 2x$, $x \geq 0$, and the second one — to the ray $x = 0$, $y \leq 0$.

B7. Arguing, as in Problem B3d, we see that the intersection of zero set with the first quadrant, is symmetric with respect to abscissa axis to the intersection of zero set with the fourth quadrant, and is symmetric with respect to coordinate beginning to the intersection with third quadrant.

B8. In the third quadrant $x < 0$, $y < 0$. It means that all monomials in F_N are positive, so the equality $F_N(x, y) = 0$ is impossible.

B9. (a) The solution is analogous to the solution of Problem B6a.

(b) It follows from the problem B3d, that the intersection of a set of zeros with the second quadrante lies near sets $x^2 = y \geq -x$, $x^2 = -x \geq y$, $-x = y \geq x^2$. Signs of monomials $-x^N$ and x^{2N} coincide in the second quadrant. So the intersection of a set of zeros with the second quadrant lie only near the first and the third of the named sets.

B10. Arguing, as in Problem B9b, we see that the intersection we study lies near the unite of sets $(1, y)$, $0 \geq y \geq -1$ and $0 \leq x = -y \leq 1$.

B11. Any edge of a tropical curve may be defined by the system which consists of one equation and some inequalities of type$ix + jy + b_{ij} = kx + ly + b_{kl} \geq px + qy + b_{pq}$. If this system is compatible, then the equation defines some line, and inequalities show that one must take a ray or segment instead of all line.

B12. *Hint.* Really, let us study zeros of a polynomial $F_N(x, y)$ in each quadrant separately. The map $LOG: (\mathbb{R} - \{0\})^2 \to \mathbb{R}^2$, $(x, y) \mapsto (\log_2 |x|, \log_2 |y|)$ is a bijection of each quadrant to the plane. Let us take any quadrant (for example $x, y > 0$), and let us identify it with the plane with this map. A tropical curve, corresponding to a tropical polynomial $\max\limits_{i+j \leq d}\{ix + jy + b_{ij}\}$, $b_{ij} = \log_2 |a_{ij}|$, divides a tropical plane to some areas. In each of this areas one of the monomials $(a_{ij} x_i y_j)^N$ defines the behavior of the polynomial $F_N(x, y)$, and it is positive or negative, correspondingly to the sign of a coefficients a_{ij} (of course, it depends also from the quadrant). Let us paint each area of the complement to tropical curve in one color, if F_N is positive in this area, and in other color, if F_N is negative in it. If two areas i-th common edge are painted in different colors, then, by the Theorem on intermediate value near this edge lies some branch of the set of zeros of F_N. Now, if such two areas are painted in the same color, then no real point of the curve lies near this edge. So, for big odd values of N the set of zeros of F_N (in chosen quadrant) may be approximately shown as a set of some edges of a tropical curve (which may be implicitly named), and the set of zeros of F_N in all the plane is, approximately, a Viro curve.

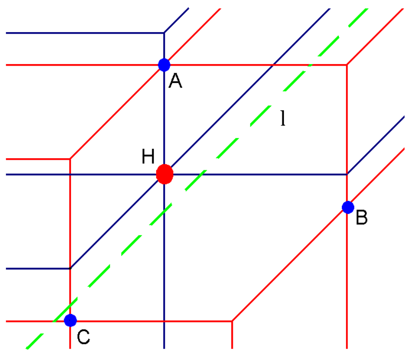
Principally, the set of zeros of F_N could have more branches, than Viro curve — for example, there could be some "small" ovals near vertices of tropical curve. But the supposition about the number of Viro curve ovals (we suppose it has $(d-1)(d-2)/2 + 1$ ovals), combined with Harnak theorem guarantees us from superfluous branches and ovals.

Remark. Authors of the problem don't know, is Viro patchworking theorem is true without the supposition that Viro curve has $(d-1)(d-2)/2 + 1$ ovals.

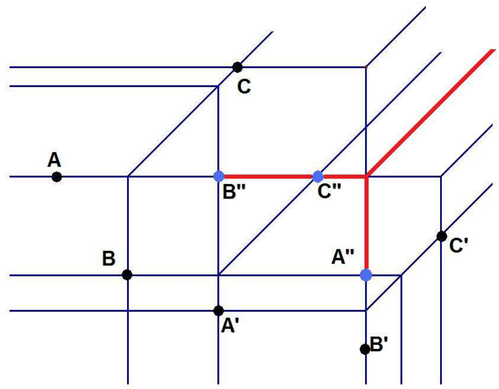
C5. (a) See figure 5.a.

(b) See figure 5.b.

(c) See figure 6.



a



b

Figure 5.

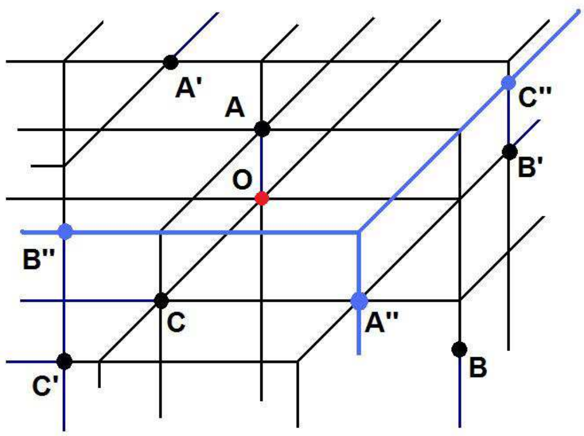


Figure 6.

D1. (a) *Hint.* The behavior of a function $\max\{x+a, y+b, c\}$ is such one. If x and y are negative and big (by module) then the constanta c is the biggest of our three values. When x increases, nothing changes till the point (x, y) will intersect the vertical line $x+a=c$. After such intersection the value $x+a$ is maximal. Similarly, when the point (x, y) moves up, maximal value is still c till it would reach the horizontal line $y+b=c$. On it both values $y+b$ and c are maximal, later - only $y+b$. At last, the areas, in which maximal value is $x+a$ or $y+b$, are divided by the ray of the line $x+a=y+b$. All three rays have the common point $(c-a, c-b)$

D2. (a) It is obvious.

(b) *Hint.* Let us take any vertice of the curve. Suppose that there are r areas (supplements to tropical curve) which are near this vertice, and that in these areas maximal are functions $i_1x+j_1y+b_{i_1j_1}, \ldots, i_rx+j_ry+b_{i_rj_r}$, respectively (we suppose that the areas are numerated in positive direction, against the clock needle). Then the equality is obvious:

$$\binom{i_2-i_1}{j_2-j_1}+\cdots+\binom{i_r-i_{r-1}}{j_r-j_{r-1}}+\binom{i_1-i_r}{j_1-j_r}=0.$$

Now one has to notice only, that the vector $\binom{i_{s+1}-i_s}{j_{s+1}-j_s}$ differs from the vector v_s in the "balance condition" only by turn on 90°.

(c) *Hint.* Let us prove, for example, that the tropical curve of degree d has exactly d horizontal rays (counting with multiplicity, of course). Let us study only the part of the plane, where coordinate x is negative and very big by module. Obviously in this part only one of the values $jy+a_{0j}$, $j=0,1,\ldots,d$ may be the maximal one. It is obvious also, that in this part, when y is negative and big by module, then a_00 is maximal, and when y is positive and big by module, then $dy + a_{0d}$ is maximal. Let y grow, and suppose, that maximal value will be (successively) a_{00}, $j_1y + a_{0j_1}$, $j_2y + a_{0j_2}$, ..., $j_ky + a_{0j_k}$, $dy + a_{0d}$. One easily sees, that $0 < j_1 < j_2 < \cdots < j_k < d$. Then multiplicities of horizontal edges are equal to j_1, $j_2 - j_1$, ..., $d - j_k$. So their number (counting with multiplicity) equals $(j_1) + (j_2 - j_1) + \cdots + (d - j_k) = d$.

D3. (ab) *Hint.* Really, suppose, that in some area a tropical polynomial coincides with a linear function $ix + jy + b_{ij}$. Consider the line, which contains a segment of boundary of this area; let $px + qy + r = 0$ be its equation. Then in the neighbor area (which borders with the first one by the segment) our polynomial coincides with linear function $(i + p)x + (j + q)y + (b_{ij} + r)$. In other words, we set the equality $b_{i+p,j+q} = b_{i,j} + r$. Proceeding in the same way, we'll restore all the polynomial, area by area, by induction. The "balance condition" guarantees, that we'll never come to contradiction. The condition on behavior of tropical curve on infinity guarantees existence only such "tropical monoms", which we have got in the process, which are only possible for tropical polynomials of given degree.

D4. *Hint.* A tropical curve of degree two one may get, as usual hyperbola, by little stirring of unite of two tropical lines. Unite of two tropical lines may be defined by the sum of two tropical polynomials of degree one. A graph — set of break points of such sum — has a vertice of 4 valency, in it maximal are four functions at once. When we'll stir one of these function (small stirring), a point of 4 valency will break on two points of 3 valency. Some of such possible tropical curves of degree two are given on Figure 7.

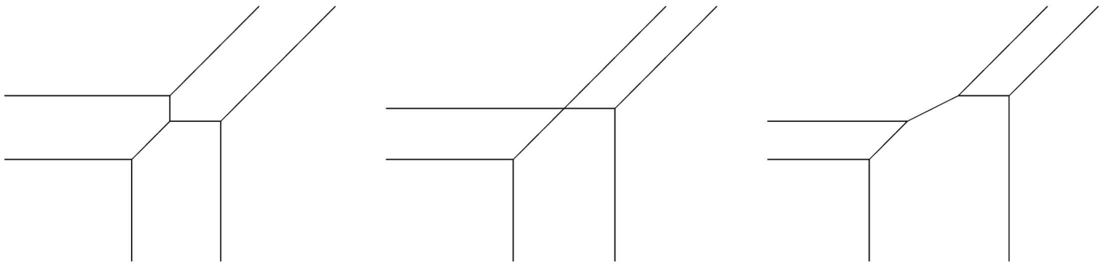


Figure 7.

D5. *Answer.* (a) 1; (b) 2; (c) 4; (d) 7.

D7. (b) **Dividing of Newton diagram.** When solving problems of part D7 it may be useful to remember such a "dual" description of tropical curves configurations. Consider a triangle on the plane whose vertices are $(0; 0)$, $(0, d)$ and $(d, 0)$. This triangle is called it a Newton triangle of a tropical polynomial. If you have any tropical curve, you have also the corresponding division of Newton triangle to a number of convex polygons with integer vertices. Namely, consider an area in complement of a tropical curve, in which the value $ix + jy + b_{ij}$ is maximal. We'll juxtapose to it a vertex with coordinates (i, j) on Newton triangle. If some edge divides two areas, we'll juxtapose to it the segment in Newton diagram from one vertex to other. At last, any vertex of tropical curve, in which r areas meet, corresponds the polygon with r corresponding vertices. In particular, if some area is infinite, the corresponding point will lie on the border of diagram, and if the edge is infinite — the corresponding segment lies on the border. It is useful to remember, that the direction of any edge of tropical curve is orthogonal to the direction of "dual" edge on diagram.

An algorithm of drawing of Viro curves. It is convenient to reformulate the procedure of drawing of Viro curves on the "dual" language of Newton diagrams. This procedure, named "Viro patchworking", consists in such successive steps (look the result in figure 8).

1. Take any triangulation of Newton diagram Δ with integer vertices;
2. In vertices of this triangulation we pose signs + or -, in arbitrary way.
3. Reflecting the Newton diagram with its triangulation respectively from coordinate axes, we get the triangulation of a square $|i| + |j| \le d$, (this square has the name of it expanded Newton diagram).
4. Now we continue posing signs on vertices of the expanded Newton diagram, as follows: sign of vertice (e_1i, e_2j) differs from the sign of vertice (i, j) by the factor $e_1^ie_2^j$, where $e_1, e_2 = \pm 1$.
5. In every triangle of our triangulation of expanded Newton diagram we'll join by a segment midpoints of edges, on whose ends signs are different (if one has such edges). Unite of all this segments is a broken line on expanded Newton diagram. This line is a combinatorial model of Viro curve.

6. Let us identify the opposite points of the border of expanded Newton diagram. Then some branches of combinatorial model of Viro curve will patch in it ovals.

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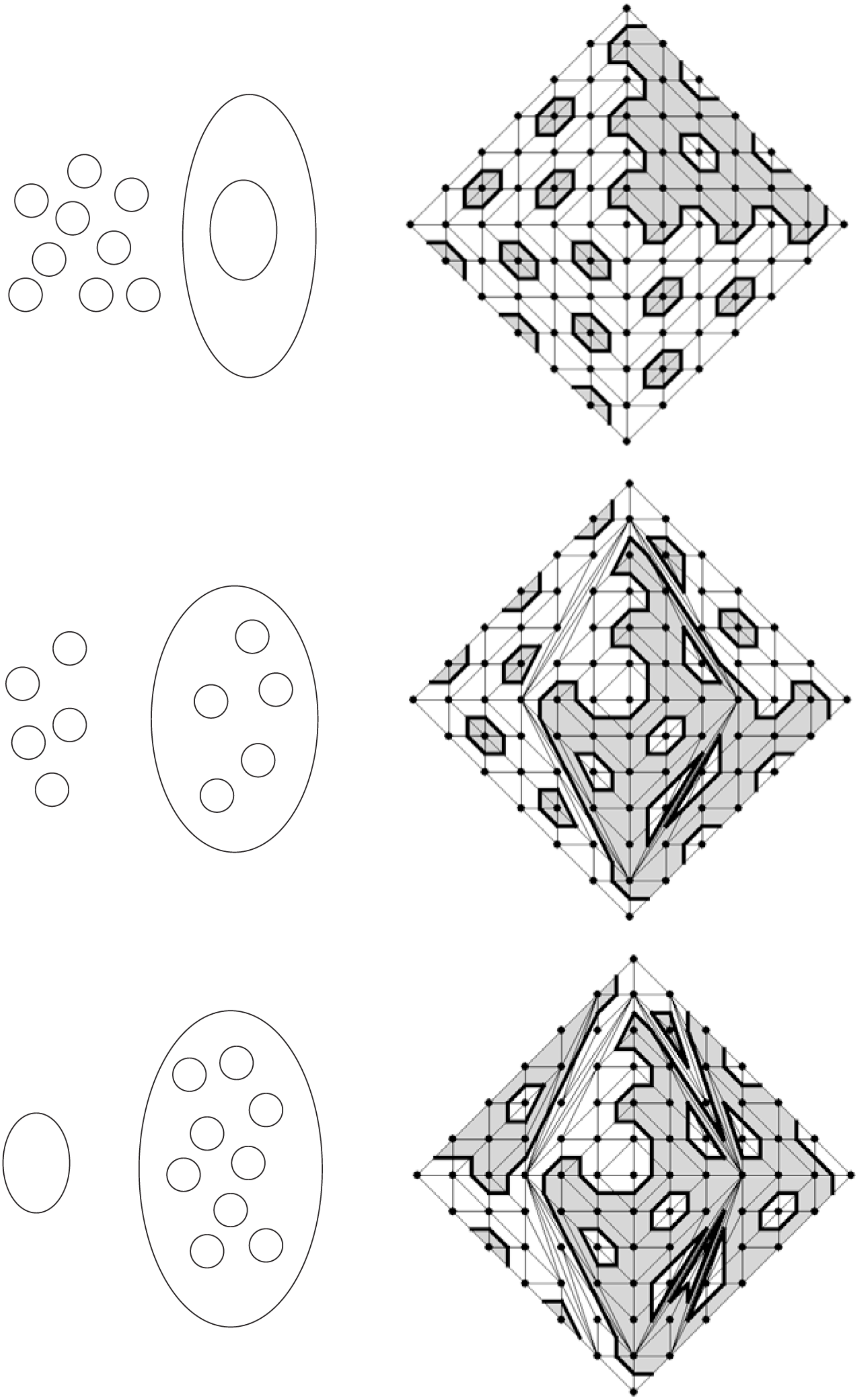

Figure 8.