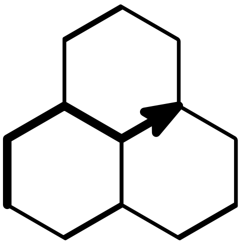

Pavements, colorings and tiling groups

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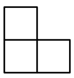
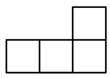
The problems of tiling often act as the centre of various mathematical matters. Very often such problems are solved with the help of coloring. One of the aims of this project is to study a more powerful method related to application of the notion of the group theory. Conjointly we will explore what, in essence, is the coloring in the new terms, and will also see how the tilings could be used in the group theory.

The first cycle is preliminary, it contains a few tiling related problems. The second and the third cycles are preparatory, during those we develop the necessary technique of the new method. During the second cycle some useful terms are introduced, and the relations between the words and the paths on graphs are studied. During the third cycle we introduce the basic terms of the group theory. In the fourth cycle we employ the new technique to the tiling problems.

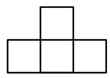
Section A. Tilings and colorings



3-cells corners



L-tetromino



T-tetromino

♦ **A1.** *Find all integers M and N such that a rectangle $M\times N$ can be cut into 3-cells corners?*

♦ **A2.** *Find all integers M, N, P such that a rectangle $M\times N$ can be cut into rectangles $P\times 1$?*

♦ **A3.** *Find all integers M, N, P such that a rectangle $M\times N$ with one additional cell can be cut into rectangles $P\times 1$?*

♦ **A4.** *Find all integers M, N, A, B such that a rectangle $M\times N$ can be cut into rectangles $A\times B$?*

♦ **A5.** *Find all integers M, N such that a rectangle $M\times N$ can be cut into L-tetromino?*

The following problem cannot be solved by colorings. We should develop some more powerful ideas to solve it. We shall do this in sections B, C, D.

♦ **A6**.** *Suppose that $M\times N$ rectangle can be tiled by T-tetramino. Prove that M and N are divisible by* 4.

Section B. Spade-work: words, graphs and paths

Let an alphabet A be a set of letters a, b, c, …. We can use these letters to arrange words which are finite sequences of these letters, for example, *abbc*, *c*, *cabcabccbb* etc. A word is called *product* of two words A and B if it is congruent with the word AB (B attached to the right of A). Hence, an alphabet determines the set of words with product operation.

A word written N times in a row ($\underbrace{XX\dots XX}_{N\text{ times}}$) is called N*-power* of X and is denoted by X^N.

♦ **B1.** *Suppose that the product UV is congruent to the product VU. Prove that the words U and V are powers of some word A.*

Consider a set of elements with some product operation $*$ (the product of two elements of the set also belongs to the set). This set is called a *semigroup* with respect to the $*$ operation if $\boldsymbol{a} * (\boldsymbol{b} * \boldsymbol{c}) = (\boldsymbol{a} * \boldsymbol{b}) * \boldsymbol{c}$ holds for all elements $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. This is *associative* property.

Examples. The set of integers is a semigroup with respect to the sum operation. The set of rational numbers is a semigroup with respect to the product operation. In this example we cannot change product by division because of that we cannot divide by zero.

The set of words determined by an alphabet is a semigroup with respect to the operation of attaching of one word to another. (It is clear that associative property holds.) More precisely, it is called *free* semigroup.

♦ **B2.** *Peter and John assembly various words using the letters* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$. *One can delete any of the neighboring pairs (any order)* $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ *from any word. Also, one can add any of these pairs into any part of any word. For example, we can transform the word* ***dacdbeaaf*** *by the following way:* ***dacdbeaaf*** $\to$ ***dabeaaf*** $\to$ ***deaaf*** $\to$ ***dfeeaaf***. *Prove that any word can be transformed to the form containing the minimal number of letters. Prove that this form doesn't depend on a set of operations applied.*

Consider a polimino. Let us mark the edges on the boundary of this polimino with arrows placed clockwise (together) or counter-clockwise. Let us associate any polimino with the sequence of the letters on its boundary. For "up", "down", "right", "left" arrows we shall accordingly write $\boldsymbol{U}$, $\boldsymbol{D}$, $\boldsymbol{R}$, $\boldsymbol{L}$ letters. At the same time, the sequence of the letters in the word must correspond to the sequence of boundary arrows. Hence, there are several words correspond to each polimino. They can be transformed one to another by the cyclic shift.

Figure 1.

♦ **B3.** *Describe the set of words corresponding to two-cells domino and three-cells corner trimino. (You should take into account various placements of the poliminoes on the plane.)*

♦ **B4.** *Let* $\boldsymbol{M}$ *be the set of words corresponding to the various placements of the two-cells domino on the plane. Suppose that we can apply the following operations to any word in* $\boldsymbol{M}$:
1. Add or delete neighboring pairs $\boldsymbol{R}$ and $\boldsymbol{L}$ *или* $\boldsymbol{U}$ and $\boldsymbol{D}$ *(in any order, as in B2);*
2. Add $\boldsymbol{R}$ *letter to the beginning of the word and* $\boldsymbol{L}$ *letter to the end of the word (and vice versa);*
3. Add $\boldsymbol{U}$ *letter to the beginning of the word and* $\boldsymbol{D}$ *letter to the end of the word (and vice versa);*
4. Attach obtained words to each other.

Let some region can be tiled by dominoes. Prove that the word corresponding to the boundary can be obtained by complex of these operations.

It is easy to see that: operation 1 corresponds to adding or deleting of pair of «there and back again» paths; operation 2 and 3 corresponds to the changing word by its cyclic shift; operation 4 corresponds to the attaching words to each other (using the fact that we can delete «there and back again» paths). Thus, we can formulate the necessary tiling condition using our language of word products.

♦ **B5.** *Suppose that we can tile the fixed finite region using the set of poliminoes* $\boldsymbol{T}$. *Consider the original set of words corresponding to* $\boldsymbol{T}$. *Prove that the word corresponding to the boundary is presented by the result of the several product and cyclic-shift operations.*

♦ **B6.** *Construct an example of the region which boundary can be obtained by the operations of* $\boldsymbol{B5}$, *but tiling cannot be done.*

Consider a polygon on the plane. Suppose that it is cut on N small polygons such that none of vertices of the small polygons belong to the inner part of other polygon edge. Consider closed paths of arbitrary length passing through the small polygons edges. Suppose that adding (or delete) the "there and back" edges doesn't change any path. Let point A be some vertex of a small polygon. If the path can be presented by pass through one path then do through the second one, then we say that our path is a product of two passed paths.

♦ **B7.** *Consider an infinite set of closed paths that start at point A. Prove that there exists a finite set P of closed paths from A such that any closed path starting at A can be presented by the finite product of the paths from P. Find the minimal number of paths in the set P.*

♦ **B8.** *Choose another vertex B. Consider closed paths starting at B. Prove that we can provide a correspondence such that the following properties hold:*

1. Every path starting at A corresponds to its own path staring at B.

2. If paths p_1 and p_2 starting at A correspond to the paths p'_1 and p'_2, then product p_1p_2 corresponds to the path $p'_1p'_2$.

Section C. Groups

A set with respect to the product operation $*$ is called *a group* if the following conditions hold:

1. Associative property. For any a, b, c holds $a * (b * c) = (a * b) * c$.
2. The existence of the unit element. There exists an element e such that the equality $ae = ea = a$ holds for any a in the group.
3. The existence of the inverse element. For any a in the group there exists an element a^{-1} such that $aa^{-1} = a^{-1}a = e$.

Examples. The set of integers is a group with respect to the sum operation. An opposite number is an inverse element. The set of rational numbers (without zero) is a group with respect to the product operation.

Substitution groups. There are six ways to place three elements in the row: 213, 321, 132, 312, 231, 123. In fact, we consider the transformations of the three elements: we can interchange two of them (so we obtain 213, 321 or 132), or we can shift all three elements by the cycle (so we obtain 312 or 231). Also we can do nothing so we stay with 123 in this case. It is easy to see that we have six different «actions». If we apply one of these actions and immediately another one, then we obtain the action from the six ones again. For example, if we interchange the first two elements and shift all three elements by the cycle to the right, then we obtain $123 \to 213 \to 321$. Thus, we obtain the action of interchanging of the first and third elements. These «actions» are called *substitutions*. The set of substitutions is a group with respect to the consecutive applying operation. There are six elements in this group. The identical transformation 123 is the unit element. This group is the smallest group such that there holds $ab \neq ba$ for some a and b. We denote it by S_3. We can consider substitution group S_n for any integer n.

♦ **C0.** *Check that paths in the B8 and B9 form a group.*

A group is called *abelian* or *commutative* if $ab = ba$ holds for any a and b.

♦ **C1.** *Construct a nonabelian group consisting of 8 elements.*

♦ **C2.** *Prove that the set of motions in space that transfer the cube to itself is a group. Find the number of elements in this group.*

♦ **C3.** *Prove that the set of words in the alphabet $\{a_1, {a_1}^{-1}, a_2, {a_2}^{-1}, \ldots, a_n, {a_n}^{-1}\}$ is a group with respect to the attaching operation. (Also we can use cancellation of neighboring inverse letters x and x^{-1}.)*

Let G be a group. Suppose that H is a subset of G and if $a, b \in H$, then $a^{-1}, b^{-1}, ba, ab \in H$. Then we say that H is *a subgroup* of G. H is a group too, with respect to the same operation.

♦ **C4.** *Prove that every element in the substitution group S_n is a product of several independent cycles. Describe all subgroups in the substitution group S_5.*

Elements a and b are called *conjugate* if there exists an element x, such that $xax^{-1} = b$. If H is a subgroup, then *conjugate set* xHx^{-1} is also a subgroup, because of $xh_1x^{-1}xh_2x^{-1} = xh_1h_2x^{-1}$.

♦ **C5.** *Find all conjugate subgroups in the S_5 substitution group.*

A group often can be described by different methods. For example, elements of a group can be presented by the motions in space transferring polyhedron to itself, or by substitutions of several elements. The structure of a group be the same. There is a special notion to describe the «sameness» idea.

Groups G and H are called *isomorphic* if there exists a correspondence between G and H with the following properties:

1. There is an unique element $h \in H$ corresponding to any element $g \in G$, and vice versa;

2. If elements $h_1, h_2 \in H$ correspond to the elements $g_1, g_2 \in G$, then product h_1h_2 corresponds to g_1g_2.

♦ **C6.** *Suppose that G is a group of motions transferring the fixed cube to itself. Find a substitution group which isomorphic to G.*

Suppose that the elements of a group are colored in several colors such that the color of product depends on colors of the elements only and not depends on the choose of elements of that color. Hence, the product of elements colored in 1 and 2 colors always has the same color (we can take any elements with color 1 and 2).

♦ **C7.** *Prove that the set of all elements having the same color as unit element is a subgroup. This subgroup is called normal.*

Equivalent definition. A subgroup $H \in G$ is called *normal* if for any $g \in G$ (g may be not in H) the following holds: $gHg^{-1} \in H$. In other words, a normal subgroup is conjugate to itself.

♦ **C8.** *Prove this equivalence of the definitions.*

♦ **C9.** *Find all normal subgroups in the substitution group S_4.*

♦ **C10.** *Consider a group G. Let K be a set of elements $aba^{-1}b^{-1}$. (they are called commutators). Suppose that H is the set of all possible products of elements in K. Prove that H is a normal subgroup.*

Section D. Tiling groups

A set of cells on the square lattice is called *connected* if we can walk from any cell to another by passing thought cell edges.

Let T be a set of cells on the square lattice. Suppose that T have connected complement set. If we can walk through the boundary of T passing all the edges one time only, then we say that T is a *tile*. Now we shall consider the pavements with tiles.

Consider a set of words corresponding to the finite tile set. Let us play with this set the game as in B4 problem: add or delete pairs of inverse letters, or change word by its conjugate, or attach words to each other (take words products).

♦ **D1.** *Prove that the set of all words that we obtain by such operations is a group with respect to the product.*

This group is called *a tiling group* of the given tile set.

♦ **D2.** *Let C be a set of all closed paths on the square lattice. Prove that the set of words corresponding to these paths is a group. Prove that for any tile set a tiling group is a normal subgroup in C group.*

♦ **D3.** *Suppose that region O can be tiled by tile set T. Prove that the word dO corresponding to the boundary of O is in the tiling group of T.*