Three parabolas
Solutions
F.Nilov, A.Zaslavsky

2 Introductory problems

1. Let X' be the projection of X to the directrix. Then $FX = XX'$. Suppose that the bisectrix l of the angle $X'XF$ isn't the tangent to parabola. Then l intersect the parabola in some point Y, distinct from X. Note that the triangles FXY and $X'XY$ are equal and so $FY = YX'$. Let Y' be the projection of Y to the directrix. Then $YY' = FY = YX'$. But it is impossible because YY' and YX' are the hypothenuse and the cathetus of rectangle triangle $YX'Y'$.

2. Let l touche the parabola in the point X. Then $FX = XX'$. By problem 1 l is the bisectrix of isosceles triangle FXX'. So l is the medial perpendicular to the segment FX', and the points X' and F are symmetric wrt l.

3. By problem 2 the lines PX and PY are medial perpendiculars of segments FX' и FY'. So their common point P is the circumcenter of $FX'Y'$.

4. Consider the medial line of the trapezoid $XX'Y'Y$. It is perpendicular to the directrix and so parallel to the axis of the parabola. Also it is the median of PXY, because it pass through the midpoint of segment XY and (by problem 3) through P.

5. Consider the line l passing through P and parallel to the axis of the parabola. We must prove that the angle ϕ between l and PX is equal to the angle FPY. Let X'' and Y'' be the projections of F to the lines PX and PY. By problem 2 X'' and Y'' are the midpoints of FX' and FY'. So the lines $X'Y'$ and $X''Y''$ are parallel. This yields that l and $X''Y''$ are perpendicular. Now we have $\angle\phi = 90^\circ - \angle PX''Y'' = 90^\circ - \angle PFY'' = \angle FPY$, q.e.d.

6. Let P and Q be isogonal conjugated. Note as P_c and Q_c, P_a and Q_a, P_b and Q_b the reflections of P and Q in AB, BC and AC. It is evident that $BP_c = BP = BP_a$ and $\angle P_cBP_a = 2\angle B$. Note that $\angle QBC = \angle PBA = \angle ABP_c$. So $\angle P_cBQ = \angle B = 1/2\angle P_cBP_a = \angle P_aBQ$, and $QP_c = QP_a$. Similarly $QP_a = QP_b$ и $PQ_c = PQ_a = PQ_b$. Using the homothety with center P and coefficient $1/2$ we receive that the circle ω with center in the midpoint of PQ and the radius $QP_c/2$ is the pedal circle of P. Similarly ω is the pedal circle of Q.

7. As Π_a and Π_b are inscribed in the angles A and B their common points lie inside or on the sidelines of the triangle ABC. C isn't the unique common point of parabolas because their tangents in C doesn't coincide. So there exists the common point C' distonct from C. It is clear that the sidelines of ABC doesn't contain the common points distinct from C. Suppose that Π_a and Π_b have inside the triangle the common point C'' distinct from C'. As A, C, C' и C'' lie on Π_b, they are the vertex of convex quadrilateral. Similarly B, C, C', and C'' are the vertex of convex quadrilateral. But it is impossible. So Π_a and Π_b have exactly two common points

8. It is known that an arbitrary affine map transforms any parabola to the parabola. Consider the map transforming the triangle ABC to regular triangle. It is evident that in regular

triangle the common points of parabolas lie on the medians. So AA', BB' and CC' are the medians of ABC and concur in its centroid M.

9. By problems 4 and 5 the lines AM and AF_a are symmetric wrt the bisectrix of angle A. So AF_a, BF_b and CF_c concur in the Lemoine point L of ABC.

10. Let the directrix d_a of Π_a intersect BC in the point A'. Note the projections of B and C to d_a as B_1 and C_1. It is clear that $BB_1 = BF_a$ and $CC_1 = CF_a$. As the triangles $A'BB_1$ and $A'CC_1$ are similar $\frac{A'B}{A'C} = \frac{BB_1}{CC_1} = \frac{BF_a}{CF_a}$. By problems 1 and 5 $\angle F_aBA = \angle F_aAC$ and $\angle F_aCA = \angle F_aAB$. Usung the sinus theorem for the triangle F_aBA we receive that $BF_a = \frac{\sin(\angle F_aAB)\cdot AF_a}{\sin(\angle ABF_a)}$. Similarly $CF_a = \frac{\sin(\angle F_aAC)\cdot AF_a}{\sin(\angle ACF_a)}$. So $\frac{BF_a}{CF_a} = \frac{\sin^2(\angle F_aAB)}{\sin^2(\angle F_aAC)}$.

    Let d_b and d_c be the directrix of Π_b and Π_c. Note as B', C' the common points of d_b and AC, d_c and AB. Then

$$\frac{A'B \cdot B'C \cdot C'A}{A'C \cdot B'A \cdot C'B} = \frac{BF_a \cdot CF_b \cdot AF_c}{CF_a \cdot AF_b \cdot BF_c} = \frac{\sin^2(\angle F_aAB) \cdot \sin^2(\angle F_bBC) \cdot \sin^2(\angle F_cCB)}{\sin^2(\angle F_aAC) \cdot \sin^2(\angle F_bBA) \cdot \sin^2(\angle F_cCA)} = 1.$$

    The last equality follows from the Ceva theorem for the cevians AF_a, BF_b and CF_c. By Menelaus theorem A', B' and C' are collinear. So by Desargues theorem the triangles are perspective.

Three parabolas
Solutions
F.Nilov, A.Zaslavsky

3 Basic problems

11. Note the midpoint of AB as C_0. We have $\angle C_0CB = \angle F_cCA$. By problems 1, 4 and 5 $\angle MCB = \angle F_cBC$ и $\angle MCA = \angle F_cAC$. So $\angle AF_cC = 180^\circ - \angle F_cAC - \angle F_cCA = 180^\circ - \angle C = 180^\circ - \angle F_cCB - \angle F_cBC = \angle BF_cC$.

12. By problem 11 $\angle AF_cC = \angle BF_cC = 180^\circ - \angle C$. So $\angle AF_cB = 360^\circ - 2(180^\circ - \angle C) = 2\angle C = \angle AOB$. This follows that A, B, F_c and O lie on the circle ω.

13. Let A_0 и B_0 be the midpoints of BC and AC. The problem 5 yields next

    Lemma. The focus of the parabola inscribed in the triangle lies on the circumcircle of this triangle

    Prove that the circumcircle of A_0B_0C pass through F_c. Note as C' the second common point of CF_c with the cirumcircle of ABC. By problems 1 and 5 $\angle F_cBC = \angle F_cCA$ and $\angle F_cAC = \angle F_cCB$. So $\angle C'BA = \angle F_cCA = \angle F_cBC$ и $\angle BAC' = \angle F_cCB$. From this the triangles F_cCB and $C'AB$ are similar and $\frac{CF_c}{C'A} = \frac{BF}{BC'}$. By problem 11 $\angle BF_cC' = \angle AF_cC'$. Also $\angle C'BF_c = \angle ABC = \angle AC'F_c$. So the triangles F_cBC' and $F_cC'A$ are similar and $\frac{F_cC'}{C'A} = \frac{BF}{BC'} = \frac{CF_c}{C'A}$. It follows that $F_cC = F_cC'$. The homothety with center C and coefficient $\frac{1}{2}$ transforms B, A and C' to A_0, B_0 and F_c, q.e.d.

14. By problem 11 the points A, B, F_c and O lie on the circle ω. Let O' be the second common point of CF_c and ω. By problem 10 $\angle AF_cO' = \angle BF_cO'$. So O' is is the midpoint of an arc AB of ω and the segment OO' is the diameter of ω. From this $\angle LF_cO = \angle OF_cO' = 90^\circ$. Similarly $\angle LF_aO = \angle LF_bO = 90^\circ$. So F_a, F_b lie F_c lie on the circle with diameter OL.

15. Let A_0 and B_0 be the midpoints of BC and AC. Consider the point F_c' symmetric to F_c wrt A_0B_0. By problem 13 A_1B_1 touches Π_c. So by problem 2 F' lies on the directrix of Π_c. By problem 13 the quadrilateral $CA_1F_cB_1$ is cyclic. So $\angle A_1F_c'B_1 = \angle A_1F_cB_1 = 180^\circ - \angle C$ and F_c' lies on the Euler circle of ABC. Note as M' the midpoint of A_0B_0. By problem 13 A_1B_1 is the bisectrix of angle CMF_c. So F_c' lies on CM'. Thus the common point of the median and Euler circle lies on the directrix of Π_c.

16. Let A'', B'', C'' be the vertex of directrix triangle. Note as A', B' and C' the common points of medians of ABC and its Euler circle. By problem 15 these points lie on the sidelines of $A''B''C''$. Note as C_0 the midpoint of AB. Let G be the centroid of $A''B''C''$. Note that

    $$\frac{\sin(\angle GC''B'')}{\sin(\angle GC''A'')} = \frac{\sin(\angle B'')}{\sin(\angle A'')} = \frac{\sin(\angle C_0MA)}{\sin(\angle C_0MB)} = \frac{MB}{MA} = \frac{MA'}{MB'} = \frac{\sin(\angle MC''B'')}{\sin(\angle MC''A'')}.$$

    The first and the third equalities are correct because $C''G$ and MC_0 are the medians of $A''B''C''$ and AMB. So $\angle GC''B'' = \angle MC''B''$ and G lies on $C''M$. Similarly G lies on $A''M$ and $B''M$. this follows that G and M coincide.

17. By problem 4 AM is perpendicular to d_a. Similarly BM is perpendicular to d_b and CM is perpendicular to d_c. So M is the orthology center of considering triangles. Let A', B' и C' be the common points of respective directrix. As the medians of ABC are perpendicular to the sidelines of $A'B'C'$ we have :

    (1) $(\overrightarrow{C'A'} + \overrightarrow{A'B'}) \cdot (\overrightarrow{AB} + \overrightarrow{AC}) = 0$.

    (2) $\overrightarrow{A'B'} \cdot (2\overrightarrow{CA} + \overrightarrow{AB}) = 0$.

    (3) $\overrightarrow{C'A'} \cdot (2\overrightarrow{AB} + \overrightarrow{CA}) = 0$.

    Summing (1) and (3) and substracting from the result (2), we receive that $\overrightarrow{CA} \cdot \overrightarrow{A'B'} = \overrightarrow{C'A'} \cdot \overrightarrow{AB}$. From this and (1) $(\overrightarrow{C'A'} - \overrightarrow{A'B'}) \cdot (\overrightarrow{AB} - \overrightarrow{AC}) = 0$. This follows that BC is perpendicular to the median $A'G$ of $A'B'C'$. By problem 16 the points G and M coincide. So $A'M$ and BC are perpendicular. Similarly $B'M$ and AC, $C'M$ and AB are perpendicular. So M is the common orthology center of ABC and $A'B'C'$.

    Note that two orthological triangles with coinciding orthology centers are perspective. So we receive another solution of problem 10.

18. As d_a is perpendicular to the median AA_0, then by problem 15 d_a pass through the common point A'' of an altitude AA' and the Euler circle. Similarly the common points of Euler circle B'', C'' of Euler circle and the altitudes BB', CC' lie on d_b and d_c. Note that Euler circle of ABC is the pedal circle of M wrt the directrix triangle. By problem 16 M is the centroid of directrix triangle. So the medians of ABC are parallel to the lines $A''L'$, $B''L'$ and $C''L'$, where L' is the Lemoine point of the directrix triangle. It is known that the triangles $A''B''C''$ and ABC are homothetic with center H and coefficient 1/2. So, as L' is the centroid of $A''B''C''$ it is the midpoint of HM and lies on the Euler line of ABC.

19. Let T be the directrix triangle. Consider the parabolas Π'_a, Π'_b и Π'_c, touching the sidelines of T in its vertex. Note as T' the triangle formed by the directrix of these parabolas. The respective sidelines of T' and ABC are perpendicular to the medians of T. So T' and ABC are homothetic and M is the homothety center because by problem 16 the centroids of T' and ABC coinside. Note the Lemoine point of T' as L'. By th homothety of T' and ABC the points M, L and L' are collinear. By problem 18 L' Lies on the Euler line of T. So the line passing through L, M and L' coincide with the Euler line of T.

20. Let C' be the common point of AB and P_aP_b. The points C', P_c, A B are harmonic because $\frac{C'A}{C'B} = \frac{AP_c}{P_cB}$. By problem 11 $\frac{AP_c}{P_cB} = \frac{AF_c}{F_cB}$. Let C'' be the common point of d_c and AB. By problem 10 $\frac{C''A}{C''B} = \frac{AF_c}{F_cB}$. So $\frac{C'A}{C'B} = \frac{AF_c}{F_cB} = \frac{C''A}{C''B}$ and the points C' and C'' coincide. By problem 10 A'', B'' and C'' are collinear. So A', B' and C' are also collinear and by the Desargues theorem the triangles ABC and $P_aP_bP_c$ are perspective.

21. By problem 20 the perspective axis of ABC, $P_aP_bP_c$ and the directrix triangle coincide. Now use next

    Lemma. If the perspective axis of three mutually perspective triangles coincide then their pair perspective centers are collinear.

    Proof. Consider the projective map transforming the commo perspective axis to the infinite line. It transforms given triangles to the homothetic triangles. Their homothtety centers are collinear.

22. Let A_1, B_1 и C_1 be the midpoints of the sides of ABC and A_2, B_2 и C_2 be the feet of its altitudes. Note as C''' the common point of CC_1 and the Euler circle. By problem 15 C''' lies on d_c. Let C' be the common point of d_c and $A'B'$, and C'' be the common point of CC_1 and A_2B_2. Note that $B_2C_1 = AC_1 = BC_1 = A_2C_1$. So $\angle B_2C'''C_1 = \angle A_2C'''C_1$, because the quadrilateral $A_2C'''B_2C_1$ is cyclic. By problem 5 d_c is perpendicular to $C'''C''$. So $C'''C'$ is the external bisectrix of angle C''', and $\frac{C'A_2}{C'B_2} = \frac{C''A_2}{C''B_2}$. Similarly $\frac{B'C_2}{B'A_2} = \frac{B''C_2}{B''A_2}$ and $\frac{A'B_2}{A'C_2} = \frac{A''B_2}{A''C_2}$, where A', B' are the common points of d_a and B_2C_2, d_b and A_2C_2; A'', B'' are the common points of AA_1, BB_1 and the Euler circle. Note that

$$\frac{C'A_2 \cdot B'C_2 \cdot A'B_2}{C'B_2 \cdot B'A_2 \cdot A'C_2} = \frac{C''A_2 \cdot B''C_2 \cdot A''B_2}{C''B_2 \cdot B''A_2 \cdot A''C_2} = 1.$$

So by Menelaus theorem A', B', C' are collinear and by Desargues theorem the directrix triangle and the orthotriangle are perspective .

23. Consider the triangle formed by the medians of ABC. By problem 5 its angles are equal to the angles of directrix triangle. Prove that the Brocard angles of ABC and the triangles formed by its medians are equal. Next formulaes are correct for any triangle:

(1) $\operatorname{ctg}\phi = \operatorname{ctg}\alpha + \operatorname{ctg}\beta + \operatorname{ctg}\gamma$.

(2) $\operatorname{ctg}\alpha + \operatorname{ctg}\beta + \operatorname{ctg}\gamma = \frac{a^2+b^2+c^2}{4S}$.

(3) $S = \frac{3}{4}S_m$.

(4) $a^2 + b^2 + c^2 = \frac{3}{4}(m_a^2 + m_b^2 + m_c^2)$.

there α, β and γ are the angles of triangle, ϕ is its Brocard angle; S and S_m are the areas of the triangle and the triangle formed by its medians; a, b and c are the lengths of the sides; m_a, m_b and m_c are the lengths of the medians.

Using these formulaes to ABC we receive

$$\operatorname{ctg}\phi = \operatorname{ctg}\alpha + \operatorname{ctg}\beta + \operatorname{ctg}\gamma = \frac{a^2+b^2+c^2}{4S} = \frac{m_a^2+m_b^2+m_c^2}{4S_m} = \operatorname{ctg}\phi_m,$$

where ϕ_m is the Brocard angle of the triangle formed by the medians. So, $\phi = \phi_m$, q.e.d.