Auxiliary conics

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It seems that the problems of p.1 are not coherent. They are assotiated by the difficulty of elementary solution. But all this problems have nice solutions using the properties of some auxiliary conics. Some theorems describing these properties are cited in p.1, the other will be given in p.2

Definition 1. *Ellipse* is the locus of points P such that the sum $PF_1 + PF_2$ of distances from P to two fixed points — *the foci* of ellipse — is the consnant.

Definition 2. *Hyperbola* is the locus of points P such that the modul of difference $|PF_1 - PF_2|$ from P to two fixed points — *the foci* of hyperbola — is the consatant. Hyperbola has two branches approaching in infinity to two lines — *the asymptotes* to hyperbola. The hyperbola with perpendicular asymptotes is called *equilateral.*

Definition 3. *Parabola* is the locus of points P such that the distances from P to the fixed point F and the fixed line l — *the focus* and *the directix* of parabola — are equal. The perpendicular from F to l is called *the axis* of parabola.

Definition 4. The points P and Q are *isogonally conjugated* with respect to the triangle ABC, if the lines AP and AQ, BP and BQ, CP and CQ are symmetric with respect to the bissectors of the respective angles.

Definition 5. Let the quadrilateral be given. The line passing through the midpoints of its diagonals is called *the Gauss line.*

1 The problems

1. Let four general lines be given.

a) Prove that the circumcircles of four triangles formed by this lines have the common point (*the Michel point*).

b) Prove that the orthocenters of these triangles lie on the line perpendicular to the Gauss line of given quadrilateral. (This line is called *the Aubert line.*)

c) (**L.Emelyanov**) Let three lines distinct from given and passing through their common points be considered. Prove that the nine point circle of the triangle formed by these lines pass through the Michel point of given quadrilateral and the circumcenter of this triangle lies on the Aubert line.

2. Given the triangle ABC and two points P, Q. The lines AP and AQ intersect BC in the points A_1, A_2 respectively. The point B_1, B_2, C_1, C_2 are defined similaply. (The triangle $A_1B_1C_1$ is called *the cevian triangle* of P with respect to thev triangle ABC.) A_3 — is the common point of AA_1 and B_2C_2; A_4 — the common point of AA_2 and B_1C_1; B_3, C_3, B_4, C_4 are defined similarly. Prove that the lines A_1A_4, A_2A_3, B_1B_4, B_2B_3, C_1C_4, C_2C_3 are concurent.

Addition. What is the common point of these lines when P and Q are:

a) the centroid and the Gergonne point (the common point of the lines passing through the vertex and the touching points of opposite sidelines with the incircle);

b) the centroid and the orthocenter;

c) two diametral points of the circumcircle.

3. Prove **the optical property** of the parabola: the tangent in point X to the parabola with th focus F formes the equal angles with XF and the axis of parabola.

4. Prove that the reflection of focus in the tangent lies on the directrix of the parabola.

5. Find the locus of projections of the focus to the tangents of the parabola.

Theorem 1. If the sidelines of the triangle touche the parabola then the circumcrcle pass through the focus and the orthocenter lies on the directrix.

Theorem 2. There exists the single conic passing trough five given general points.

Theorem 3. There exists the single conic touching five given general lines.

Theorem 4. All conics are projectively equivalent. In part any conic can be projectively transformed to the circle. This allows to define the polarity with respect to any conic and to formulate the duality principe.

6. The points X and X', Y and Y' are isogonally conjugated with respect to the triangle ABC. U, V are the common points of XY and $X'Y'$, XY' and $X'Y$. Prove that U and V are isogonally conjugated with respect to ABC.

7. The triangles ABC and $A'B'C'$ are centrosymmetric. Three parallel lines pass through A', B', C'. Prove that their common points with BC, CA, AB respectovely are collinear.

8. Each of three circles lies outside two other. The hexagon formed by their common internal tangents is considered. Prove that its main diagonals concur.

9. The distances from the point T to the opposite sidelines of the convex quadrilateral are equal. Prove that T lies on the Gauss line iff the quadrilateral is inscribed, circumscribed or the trapezoid.

10. The points A, B are inside the angle with vertex O. The billiards ball can come from A to B after the reflection from one side of the angle in the point X or after the reflection from the other side in the point Y. The points C, Z are the midpoints of AB, XY respectively.

a) $\angle O = 90^\circ$. Prove that the line CZ pass through O.

b) $\angle O \neq 90^\circ$. Prove that CZ pass through O iff lengths of path AXB and AYB are equal.

11. (**The Droz-Farny theorem**) Two perpendicular lines pass through the orthocenter of the triangle ABC. Prove that the midpoints of segments striked by these lines in the sidelines of ABC are collinear.

12. (**L.Emelyanov**) AA_1, BB_1 are the altitudes of the triangle ABC; C^* is the point on the line A_1B_1. Any line passing through C^* intersect BC and AC in the points A' and B' respectively. P is the common point of AA' and BB'; C' the common point of AB and CP. Prove that all circumcircles of the triangles $A'B'C'$ have the common point.

13. AA_1, AA_2 are the altitude and the bissector of the triangle ABC; A_3, A_4 are the touching points of BC with the incircle and the excircle. The points $B_1, \dots, B_4$, $C_1, \dots, C_4$ are defined similarly.

a) Prove that the lines A_1B_1, A_2B_2, A_3B_3, A_4B_4 concur.

b) Prove that the circumcircles of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$ have the common point.

14. Given the triangle ABC and the line passing through its circumcenter O. Prove that the pedal circles of all points on this line have the common point. (The triangle formed by the projections of the point P to the lines AB, BC, CA and its circumcircle are called *the pedal triangle* and *the pedal circle* of the point P with respect the triangle ABC.)

15. Two triangles are similar, oppositely oriented and their orthocenters coincide. Prove that they are perspective.

2 The properties of conics

Theorem 5. (**Pascal**) Six points lie on the conic iff the common points of opposite sidelines of respective hexagon are collinear.

Theorem 6. (**Brianchon**) Six lines touche the conic iff the main diagonals of respective hexagon concur.

Theorem 7. The Gauss line of the quadrilateral is the locus of the centers of inscribed conics.

Theorem 8. Let four points A, B, C, D be given. X, Y, Z are the common points of AB and CD, AC and BD, AD BC respectively. P is any point distinct from X, Y, Z. Then all polars of P with respect to the conics passing through A, B, C, D have the common point. In part if A, B, C, D are orthocentric then this point is isogonally conjugated to P with respect the triangle XYZ.

Theorem 9. Let the line l don't pass through the points A, B, C. Then the the isogonal image of l with respect to the triangle ABC is the circumconic of ABC.

Theorem 10. Let two triangles $A_1B_1C_1$ and $A_2B_2C_2$ be given. C', A', B' are the common points of A_1B_1 and A_2B_2, B_1C_1 and B_2C_2, C_1A_1 and C_2A_2 respectively. If the triangle $A'B'C'$ is perspective to both triangles $A_1B_1C_1$ and $A_2B_2C_2$ with perspective centers D_1, D_2, then the points A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 lie on the conic.

Theorem 11. (**Four conics theorem**) Let each two from three conics have four common points. Six common points (two for each pair of conics) lie on the conic iff three common chords passing through another six points concur.

Theorem 12. The tangents from the point P to the parabola are perpendicular iff P lies on the directrix.

Theorem 13. The circumconic of the triangle is the equilaterale hyperbola iff it pass through the orthocenter.

Theorem 14. The nine points circle of the triangle is the locus of centers of equilateral circumhiperbolaes.

Theorem 15. The pedal and the cevian circles of the point P with repect to the triangle ABC pass through the center of equilateral hyperbola $ABCP$.

3 References

A.V.Akopjan, A.A.Zaslavsky. The properties of conics. AMS, ???.