Coverings by cell figures.
Solutions.

2 Introductive problems

2.1. Suppose that our figure consists of cells with number 0 and n. Cover all cells of $n\mathbb{Z}$ with translates $\{2kn,(2k+1)n\}$ $(k\in\mathbb{Z})$.

All the subsets $n\mathbb{Z}+m \quad m\in[1;n-1]$ can be covered by the same way. Since each cell is covered exactly once, the non-efficiency of this covering is 1.

2.2. ***(i)*** Consider an arbitrary covering and an arbitrary translate F_1 in it, we can assume that its cells are 0,1,3. The cell with number 2 is covered by some translate F_2, obviously, F_1 and F_2 must overlap. So, each translate overlaps with some other one in the covering.

If a cell is covered k times, we say that the *significance* of this cell is $1/k$. Next, define the of a translate in a covering as the sum of the significances of all its cells.

So, the significance of any translate does not exceed $\frac{5}{2}$ (since the significance of at least one of its cells does not exceed 1/2). On the other hand, the total sum of significances of the translates covering some segment of length n is at least n, since a cell with significance $1/k$ is covered k times.

Thus, we need at least $\frac{2}{5}n$ translates to cover a segment of length n, and the non-efficiency is at least $\frac{6}{5}$.

One can cover a segment of length 5 by two translates of our figure; tiling the line by these segments, one obtains the covering of non-efficiency $\frac{6}{5}$.

(ii) Since all the cells of our figure have even numbers, the coverings of the sets of even and odd numbers can be considered independently. On each of these sets our figure looks like ■■□■■.

Consider an arbitrary covering. Each translate $\mathcal{F}_1$ overlaps with the translate covering the middle cell of $\mathcal{F}_1$. So, introducing the significance as above, one can prove that non-efficiency is at least $\frac{8}{7}$.

One can easily cover a segment of length 7 by two translates; this allows one to construct the covering of non-efficiency $\frac{8}{7}$.

(iii) This figure consists of two translates of the previous one, so any covering with our figure can also be considered as the covering by that one. The non-efficiencies of these coverings coincide, hence our figure has the non-efficiency at least $\frac{8}{7}$.

On the other hand, the set $T=7\mathbb{Z}=\{\dots,-14,-7,0,7,14,\dots\}$ gives an example of the covering with non-efficiency $\frac{8}{7}$.

2.3. For an arbitrary $\varepsilon>0$, consider a figure $\mathcal{F}$ such that $\nu(\mathcal{F})\geq a_1(n)-\varepsilon$. Take a figure $\mathcal{F}'$ consisting of two disjoint translates of $\mathcal{F}$ (e.g. $\mathcal{F}'=\mathcal{F}+\{0,d+1\}$, where d is the diameter of $\mathcal{F}$).

Then any covering by $\mathcal{F}'$ is a covering by $\mathcal{F}$ as well; these coverings have equal non-efficiences. So, the non-efficiency of any covering by $\mathcal{F}'$ is at least $a_1(n)-\varepsilon$, hence $\nu(\mathcal{F}')\geq a_1(n)-\varepsilon$. Therefore $a_1(2n)\geq a_1(n)-\varepsilon$; since ε can be arbitrarily small, we get $a_1(2n)\geq a_1(n)$.

2.4. ***(i)*** If we can tile a line with a figure, then its non-efficiency is obviously 1.

Suppose that the non-efficiency of a figure is 1. Then for any $\varepsilon>0$, there exists a covering having non-efficiency less than $1+\varepsilon$. If any segment of length N in this covering contains a cell which is covered twice, then the non-efficiency of this covering is at least $1+\frac{1}{N}$. Therefore, if $\varepsilon<\frac{1}{N}$, then there exists a segment of length N which is covered only once.

Denote by d the diameter of the figure. Choose $N > 2^d + d$ and $\varepsilon < 1/N$. We can assume that the segment $[1,N]$ is covered exactly once.

Assume that we put the translates of our covering on the line one by one from the left to the right. Consider a step when all the translates whose leftmost cells have the numbers $\le n$ has been already placed. This "semicovering" is determined by the set $T_n = T \cap (-\infty, n]$. Consider the set $Z_n = T_n + \mathcal{F}$ of all cells that have been already covered up to this moment. Clearly, $(-\infty, n] \subseteq Z_n \subseteq (-\infty, n+d]$. Hence, the situations is determined only by the pattern of covered cells in interval $(n, n+d]$.

Such a pattern will be referred to as a *tail*. Formally, a *tail* is a set $H_n = \{d \mid n+d \in Z_n\} \subseteq (0,n]$.

Consider the tails $H_0, \ldots, H_{2^d}$. There can be only 2^d different tails, hence two equal ones are found. Denote them by H_m and H_n $(n < m)$.

In this situation, we call the part $T(n,m] = T \cap (n,m]$ of our covering a *cycle*, corresponding to the tail H_n.

Obviously, the union of translates $Z(n,m] = T(n,m] + \mathcal{F}$, determined by the set $T(n,m]$ covers exactly the set $(n,m] \setminus (H_n + n) \cap (H_m + m)$. Also, the translates of this set are disjoint, since they belong to $[1,N]$. So, copying this cycle with the period $m-n$, one gets the tiling $T' = T(n,m] + (n-m)\mathbb{Z}$.

(ii) Consider a tree (i. e. a connected graph without cycles). Say that one of its vertices is a *root*. For each edge, introduce its orientation in the direction from the root. Such a construction will be referred to as the *oriented tree*.

Lemma (König). Suppose that in an oriented tree, the number of vertices is infinite, while each of them has a finite degree. Then there exists an infinite path from the root vertex.

The proof is left to the reader as an exercise.

Consider an arbitrary figure $\mathcal{F}$ on the plane. The set of translates intersecting some square with the center in the origin is called a *sprout* (of a tiling), if each cell of the square is covered exactly once. The side length of this square is the *size* of the sprout.

Obviously, if there exists a sprout of size N, then there exist sprouts of all smaller sizes.

Suppose that the non-efficiency of the figure is 1. Then there exists a covering by this figure of non-efficiency less than $1 + \frac{1}{N^2}$. This covering contains a $N \times N$ square which is covered exactly once. Moving this square to the origin, one obtains a sprout of size N. Hence, there exist the sprouts of all sizes. Hence, there are infinitely many sprouts. We say that the empty set is a sprout of size 0.

Consider the following graph. Its vertices are all the sprouts, and the sprout of size n is connected with the sprout of size $n+2$ if the latter can be obtained from the former by adding some translates. Obiously, this graph is an oriented tree. One can easily prove that each vertex has a finite degree. Hence, by the König's lemma, there exists an infinite path.

Consider a union of all sprouts of this path: take all the translates in a sprout of size 2, add to them all the translates of the next sprout, and so on. Each cell belongs to all but finite number of squares, hence it will be covered exactly once. Thus, we have constructed a tiling.

2.5. ***(i)*** See problem 2.2(i)

(ii) Let figure consist of cells 0, a and $a+b$ (i. e. the distances between the cells are a and b). If $2a+b \equiv 1 \pmod 3$, then $2b+a \equiv 2 \pmod 3$. In this case, we can interchange the variables. Hence, we can assume that $2a+b \not\equiv 1 \pmod 3$.

If $\gcd(a,b) = d > 1$, then we will cover only the cells with the numbers divisible by d. All other residues can be covered in the same way. Hence, we can assume that $\gcd(a,b) = 1$.

Put translates on the line with the step $2a+b$ (that is, consider the set $T_1 = (2a+b)\mathbb{Z} = \{\ldots, -(4a+2b), -(2a+b), 0, 2a+b, 4a+2b, \ldots\}$). Then the covered set of cells $Z_1 = T_1 + \mathcal{F}$ consists of triples $\ldots$, $\{0,a,2a\} - (2a+b)$, $\{0,a,2a\}$, $\{0,a,2a\} + (2a+b)$, $\ldots$. We will cover the line by some copies of Z_1.

Consider all cells of the set Z_1 which have one fixed residue modulo a. Since a and $2a+b$ are coprime, such cells exist for each residue; one can easily prove that they form some triples on the distance $a(2a+b)$ one from another. Shift Z_1 by $3a$, $6a$, $\ldots$, until they cover all the cells of this residue (and hence of all residues).

If $2a+b\vdots 3$, then we get a tiling. In other case, $2a+b\equiv 2 \pmod 3$, and in the last translation, each triple intersects an original triple by one cell. Hence, in the cells of our residue, on each period of length $2a+b$ there is exactly one cell which is covered twice. Hence, the non-efficiency is $\nu=\dfrac{2a+b+1}{2a+b}=1+\dfrac{1}{2a+b}$. Since $2a+b\equiv 2 \pmod 3$, we have $2a+b\geq 5$ and $\nu\leq\frac{6}{5}$, QED.

2.6. ***(i)*** Delete an arbitrary cell from our figure, and then cover the line by the remaining three-cell figure with the non-efficiency $\leq]frac65$. Adding the thrown cell, we will obtain the covering of non-efficiency not greater than $\frac{4}{3}\cdot\frac{6}{5}=\frac{8}{5}$.

(ii) The estimate can be sharp only if every 3 cells of our figure form the figure of non-efficiency $\frac{6}{5}$ (prove it!). Looking at the solution of 2.5(ii), we get that in each such figure, the distances between its cells have the form $a=da'$, $b=db'$ with $2a'+b'=5$ (or vice versa). This equation has two positive solutions $(a',b')=(2,1)$ and $(a'.b')=(1,3)$. There is only one figure (with respect to stretching) with all subfigures having this form — ■■□■■. From 2.2(ii) we see that its non-efficiency is $\frac{8}{7}<\frac{3}{2}$. Hence, the estimate is not sharp.

(iii) Unfortunately, the statement of this problem is wrong. There is no such figure.

One can show this by the same way as in previous problem. If a figure has the non-efficiency $\geq\frac{3}{2}$, then the non-efficiency of each subfigure should be at least $\frac{9}{8}$. In the same notation, it is possible only for $2a'+b'=5$ or $2a'+b'=8$. There are two types of such figure — one mentioned above, and ■□■■□■. One can cover the segment of length 7 by two translates of the latter figure; hence its non-efficiency is not greater than $\frac{8}{7}<\frac{3}{2}$.

3 Main results

3.1. ***(i)*** Let a set T determine the covering by a figure $\mathcal{F}$. We can assume that $0=\min\mathcal{F}$ (i. e. the leftmost cell of $\mathcal{F}$ has a number 0). Denote the diameter of $\mathcal{F}$ by d (thus, $\max\mathcal{F}=d$). We will use the notation introduced in 2.4.

Consider an arbitrary cycle between the identical tails H_n and H_m $(n<m)$. Naturally, the number $n-m$ is called the *length* of this cycle. Spreading the copies of this cycle periodically with the period length $n-m$, we clearly get the set $T'=T(n,m]+(n-m)\mathbb{Z}$ defining the covering. The non-efficiency of this covering is $\dfrac{|T(n,m]|}{n-m}\cdot|\mathcal{F}|$. Thus, it is natural to define the *non-efficiency* of our cycle by this formula.

If there are two cycles corresponding to the same tail, then we can join them together to obtain a new cycle. Obviously, its length is the sum of the length, while its non-efficiency lies (non-strictly) between the non-efficiencies of the original cycles.

In our covering, there exists a tail occurring infinitely many times on the positive ray, as well as the tail occurring infinitely many times on the negative ray. Then we can split the covering into three parts: the cycles on negative ray, the cycles on the positive ray, and a finite part in the middle. Suppose that all these cycles have non-efficiency $\geq\beta$. Then the average number of translates on each cycle if not less than β/n, therefore the non-efficiency of our covering is not less than β.

Recall that $\nu(\mathcal{F})=\alpha$. So, for every $\varepsilon>0$, there exists a covering of non-efficiency $<\alpha+\varepsilon$, and this covering contains a cycle of non-efficiency $<\alpha+\varepsilon$. Moreover, since there exists only finite number of possible tails, there exists a tail H such that for every $\varepsilon>0$ there exists a cycle corresponding to H with non-efficiency $<\alpha+\varepsilon$.

Now, it is easy to construct a covering with non-efficiency α. Take a cycle corresponding to H with non-efficiency $<\alpha+1$. Let us join it with cycles of non-efficiency $<\alpha+\frac{1}{2}$; if we join sufficiently many such cycles, then the non-efficiency of the resulting cycle will be less than $\alpha+\frac{1}{2}$ (we add cycles from both sides of the original one). Analogously, joining to it some cycles of non-efficiency $<\alpha+\frac{1}{3}$, we obtain a cycle with non-efficiency $<\alpha+\frac{1}{3}$, and so on. Finally, we will obtain the covering of non-efficiency $<\alpha+\frac{1}{n}$ for all n; then this non-efficiency will be exactly α.

(ii) **Lemma.** Suppose that there exists a cycle of non-efficiency ϕ. Then there also exists a cycle with length $\leq 2^d$ and non-efficiency $\leq\phi$.

Proof. Consider an arbitrary covering, and an arbitrary cycle of length $n>2^d$ in it. we can assume that this cycle is $T(0,n]$. Clearly, all the tails H_i $(0<i\leq n)$ depend only on this cycle, but not on the other elements of T. Among these tails, there are two identical ones, say $H_p=H_q$, $0<p<q\leq n$.

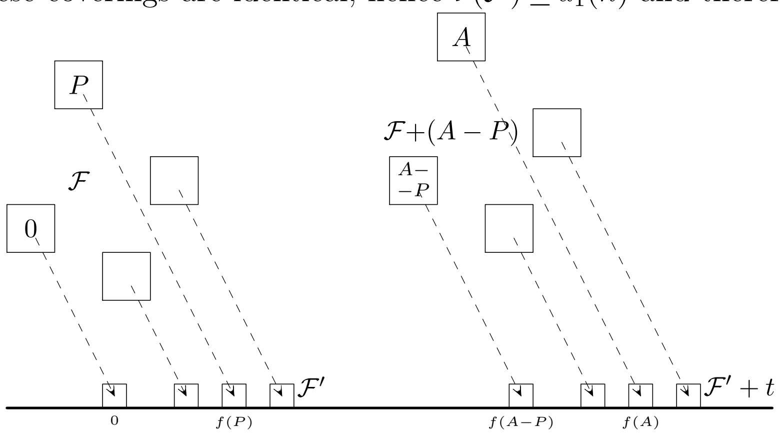
Join a piece $T(0,p]$ together with $T(q,n]$ shifted by $q-p$ to the left (that is, we consider the set $T(0,p]\cup\big(T(q,n]-(q-p)\big)$); the joint piece is a cycle of length $n-(q-p)$. Denote the non-efficiencies of this cycle and of the cycle $T(p,q]$ by β and γ, respectively. Then the non-efficiency of the original cycle $T(0,n]$ is $\dfrac{(q-p)\gamma+(n-(q-p))\beta}{n}\geq\min\{\beta,\gamma\}$, and thus one of the numbers β, γ is not greater than ϕ. Therefore, we get the cycle of the smaller length and of the non-efficiency not exceeding ϕ. Repeating this procedure, we finally get the cycle of length $\leq 2^d$, as required. □

It is easy now to finish the solution. Consider all possible cycles of the length $\leq 2^d$; there is only finite number of them. Let ν be the smallest non-efficiency of such cycle. Take an arbitrary $\varepsilon>0$. From the aforementioned, we get that there is a cycle of non-efficiency $<\alpha+\varepsilon$. Applying the Lemma, we find also the cycle of not greater non-efficiency, while its length is not greater than 2^d. Hence, $\nu<\alpha+\varepsilon$ d=for every $\varepsilon>0$ and therefore $\nu\leq\alpha$. Since there exists a covering of non-efficiency ν, we should have $\nu\geq\alpha$, so $\nu=\alpha$, and we have found the covering having non-efficiency α.

3.2. ***(i)*** Each figure on the line can be considered also as a figure on the plane; hence, $a_1(n)\leq a_2(n)$. Thus, it is sufficient to prove that $a_2(n)\leq a_1(n)$.

Consider a figure $\mathcal{F}$ on the plane (we assume that $0\in\mathcal{F}$). Let the coordinate vectors of its cells be (x_1,y_1), ..., (x_n,y_n). There exist coprime a and b such that the function $f(x,y)=ax+by$ attains distinct values on all these vectors.

Consider a figure $\mathcal{F}'=\{ax_i+by_i\mid 1\leq i\leq n\}$ on the line. Then there exists a covering T' by this figure with the non-efficiency $\nu\leq a_1(n)$. We claim that the set $T=\{(x,y)\mid f(x,y)\in T'\}$ defines the covering by $\mathcal{F}$, and the density of T is the same as that of T'. From these facts, it follows that the non-efficiencies of these coverings are identical, hence $\nu(\mathcal{F})\leq a_1(n)$ and therefore $a_2(n)\leq a_1(n)$.



To prove the first statement, consider an arbitrary cell A on the plane. Its image $f(A)$ belongs to some translate $\mathcal{F}'+t$ with $t\in T'$. Then the cell $f(A)-t$ belongs to $\mathcal{F}'$, therefore there exists a cell $P\in\mathcal{F}$ such that $f(P)=f(A)-t$. Then consider the cell $A-P$ (the subtraction is componentwise). Clearly, $f(A-P)=f(A)-f(P)=t\in T'$, hence $A-P\in T$, and the cell A belongs to the translate $\mathcal{F}+(A-P)$. We have proved that $A\in\mathcal{F}+T$, hence T defines the covering.

Recall that we can assume T' to be periodic. Let d be the length of the period. If this period contains q elements, then $\rho(T')=q/d$. Take a large square with the side length N. It is easy to show that the number of cells A such that $f(A)$ has a fixed residue modulo d is approximately N^2/d (to be precise, this number lies between $(N-d)^2/d$ and $(N+d)^2/d$). Then the number of cells belonging to T in this square is approximately qN^2/d (more precisely, it is $qN^2/d+O(N)$); this means that $\rho(T)=q/d$, as desired.

(ii) Completely analogous.

3.3. See problem 3.6.

3.4. ***(i)*** Denote the diameter and the area of the figure by d and n, respectively $(2n\geq d)$. We will construct a periodic covering with the period length d; on each step, we will add one translate to each period (i. e., in fact we will add a series of translates which can be obtained one from another by the translations on vectors divisible by d).

We claim that by one step, we can reduce the number of the uncovered cells x on a period to at most half. Consider all d possible positions of the translate on a period. Each of the uncovered cells remains uncovered for exactly $d-n$ positions. So, the average number of the cells remaining uncovered is $x\dfrac{d-n}{d}$, hence for one of them the number of these cells is at most $x\dfrac{d-n}{d}\le\dfrac{x}{2}$.

Hence, we can reduce the number of the uncovered cells to at most half; hence, after not more than $[\log_2 d]+1$ there will be less than one uncovered cell on a period; hence, all the cells will be covered. Moreover, for d cells of a period, there are not more than $[\log_2 d]+1$ translates, and hence the non-efficiency is not more than $\dfrac{n([\log_2 d]+1)}{d}\le\log_2 d+1\le\log_2 2n+1=\log_2 n+2$.

(ii) Analogously, we obtain the estimate $\nu\le\log_{k/(k-1)}kn+1$.

3.5. ***(ii)*** Again, denote the diameter and the area of the figure by d and n, respectively ($2n\ge d$). Choose an integer $t>n$ such that $nt>d$. We will construct a periodic covering with the period length nt; as in the previous problem, on each step, we will add one translate to each period.

Suppose that on some step, there are $s>(n-1)t$ uncovered cells on the period. We claim that we can add a translate which does not intersect any previous translate. Actually, consider all possible positions of a translate on the period. each of s uncovered cells is covered by n of these translates; hence, all nt translates cover $sn>(n-1)nt$ uncovered cells; hence, one of these translates contains $>n-1$ uncovered cells, as required.

So, while there are $>(n-1)t$ uncovered cells, we can decrease their number by n on each step. Hence, after $\left\lceil\dfrac{t}{n}\right\rceil$ steps we will get the situation with $\le(n-1)t$ uncovered cells on the period.

By analogous reasons, while there are $>qt$ uncovered cells, there always exists a translate which covers at least $q+1$ of them. Applying this, we achieve $\le(n-2)t$ uncovered cells after not more than $\left\lceil\dfrac{t}{n-1}\right\rceil$ additional steps, $\le(n-3)t$ cells after not $\left\lceil\dfrac{t}{n-2}\right\rceil$ steps and so on. Proceeding in this way, we cover the whole period by not more than $\left\lceil\dfrac{t}{n}\right\rceil+\left\lceil\dfrac{t}{n-1}\right\rceil+\dots+\left\lceil\dfrac{t}{2}\right\rceil+\left\lceil\dfrac{t}{1}\right\rceil$ translates. It is known that $\dfrac{1}{n}+\dfrac{2}{n}+\dots+\dfrac{n}{n}=\ln n+O(1)$ (for instance, one can prove it using the integral $\int_1^n\frac{dx}{x}$). Hence, $\left\lceil\dfrac{t}{n}\right\rceil+\left\lceil\dfrac{t}{n-1}\right\rceil+\dots+\left\lceil\dfrac{t}{2}\right\rceil+\left\lceil\dfrac{t}{1}\right\rceil=(\ln n+O(1))t$, and we have covered the period of length nt by translates with the total area $(\ln n+O(1))nt$. Thus, the non-efficiency of this covering is $\ln n+O(1)$.

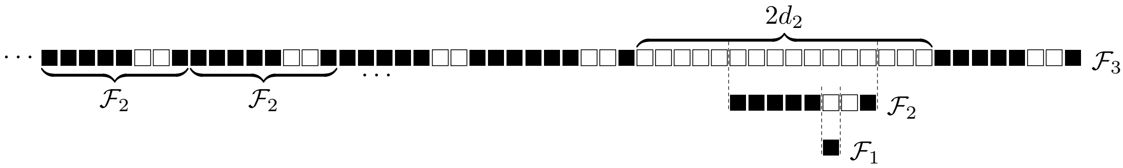
3.6. We present the series of figures $\mathcal{F}_k$ having the area $S_k=2^{k-2}(k-1)!(k+1)!$ and diameter $d_k=2^{k-1}k!^2=2\dfrac{k}{k+1}S_k<2S_k$, while $\nu(\mathcal{F}_k)>\dfrac{k}{8}$.

For $k=1$, let $\mathcal{F}_1$ be a cell. Suppose that $k>1$, and the figure $\mathcal{F}_{k-1}$ of area S_{k-1} and diameter d_{k-1} is already constructed. We construct $\mathcal{F}_k$ as follows. Put on the line in a row $2k^2-3$ translates of $\mathcal{F}_{k-1}$, and put one more its translate on the distance of $2d_{k-1}$ from them. Then, the constructed figure $\mathcal{F}_k$ has an area $2(k^2-1)S_{k-1}=S_k$ and diameter $d_{k-1}\cdot 2k^2=d_k$.

Consider a tiling by a figure $\mathcal{F}_k$. Choose any interval I of length $4d_k$; we claim that it contains at least k translates of our covering. Clearly, the translate containing the middle cell of I lies in I, and the distance between this translate and the border of I is at least d_{k-1}. We can assume that this translate is $\mathcal{F}_k$ itself.

Consider a "hole" in $\mathcal{F}_k$ of size $2d_{k-1}$. Its middle cell is covered by some translate $\mathcal{F}_k+t_k$ in our covering. It is easy to show that $\mathcal{F}_k+t_k$ contains a translate of $\mathcal{F}_{k-1}$ lying in our "hole". Since $\mathcal{F}_k$ consists of some translates of $\mathcal{F}_{k-1}$, we can consider our covering as the covering by the translates of $\mathcal{F}_{k-1}$. Then, inside a "hole" in the chosen translate of $\mathcal{F}_{k-1}$, one can find a translate of $\mathcal{F}_{k-2}$, and so on. Obviously, all the translates chosen are contained in k distinct translates of $\mathcal{F}_k$ (since each translate lies inside a hole in the pervious one).

Thus, the "hole" in our $\mathcal{F}_k$ intersects with $k-1$ other translates in our covering. All these translates are contained in I.



Hence, each interval with length $4d_k$ contains at least k figures, hence the non-efficiency of $\mathcal{F}_k$ is at least $\dfrac{kS_k}{4d_k} > \dfrac{k}{8}$.

Hence, we have constructed the figure $\mathcal{F}_k$ of area $S_k = 2^{k-2}(k-1)!(k+1)!$ and non-efficiency $\geq \nu(\mathcal{F}_k) \geq \dfrac{k}{8}$. Now, for every $n > e^{100}$ choose $k = \left[\dfrac{\ln n}{3\ln\ln n}\right] - 1$. Let us estimate $(k+1)!$. First, $(k+1)! \leq (k+1)^k = e^{k\ln(k+1)}$. Furthermore, $k\ln(k+1) \leq k\ln\ln n < \frac{1}{3}\ln n$, hence $(k+1)! < e^{(\ln n)/3} = \sqrt[3]{n}$. Finally, we get $S_k = 2^{k-2}(k-1)!(k+1)! \leq (k+1)!^3 \leq (\sqrt[3]{n})^3 = n$, so the area of $\mathcal{F}_k$ does not exceed n. From the other hand, $\nu(\mathcal{F}_k) \geq \dfrac{k}{8} \geq \dfrac{\ln n}{48\ln\ln n}$.

Note. The estimates in the last paragraph are **not** too raw. Actually, $S_k \geq (k+1)! \geq (k/2)^{k/2} \geq e^{k\ln k}$, hence from $k\ln k \geq \dfrac{\ln n}{6\ln\ln n}\dfrac{\ln\ln n}{2} \geq \dfrac{\ln n}{12}$ it follows that $S_k \geq \sqrt[12]{n}$.