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Around of Feet of Bisectors

Introduction

Solutions

1. Let I_2 be the B-excenter (fig. 1*a*). Let us consider the circle with diameter II_2. The vertices A and C lie on this circle, therefore its center lies on the perpendicular bisector of AC which intersects the diameter II_2 at the midarc B_0 of AC of Ω. Hence B_0 is equidistant from A, C, I and I_2.

Let I_1 and I_3 be the A-excenter and C-excenter (fig. 1*b*). Let us consider the circle with diameter I_1I_3. The vertices A and C lie on this circle, therefore its center lies on the perpendicular bisector of AC which intersects the diameter I_1I_3 at the midarc B_0' of AC of Ω containing B. Hence B_0' is equidistant from A, C, I_1 and I_3.

2. Let C' be the touch point of the incircle and AB (fig. 2*a*). Power of I with respect to Ω is $IO^2 - R^2 = -BI \cdot IB_0$. The triangles BIC' and $B_0'CB_0$ are similar, therefore $BI / IC' = B_0'B_0 / B_0C$. From problem 1 it follows that $B_0C = B_0I$, hence $BI \cdot B_0I = B_0'B_0 \cdot IC' = 2R \cdot r$. Therefore $IO^2 - R^2 = -2R \cdot r$, i.e. $IO^2 = R^2 - 2R \cdot r$.

Let C' be the touch point of the excircle ω_2 and AB (fig. 2*b*). Power of I_2 with respect to Ω is $I_2O^2 - R^2 = I_2B \cdot I_2B_0$. The triangles BI_2C' and $B_0'CB_0$ are similar, therefore $BI_2 / I_2C' = B_0'B_0 / B_0C$. From problem 1 it follows that $B_0C = B_0I_2$, hence $BI_2 \cdot B_0I_2 = B_0'B_0 \cdot I_2C' = 2R \cdot r$. Therefore $I_2O^2 - R^2 = 2R \cdot r$, i.e. $I_2O^2 = R^2 + 2R \cdot r$.

3. Let $\Omega = (O, R)$ be the circumcircle and $\omega = (I, r)$ be the incircle of some triangle. From problem 2 it follows that $IO^2 = R^2 - 2R \cdot r$. Take an arbitrary point on Ω, denote it B and draw the chords BA and BC tangent to ω (fig. 3). From similarity of the triangles BIC' and $B_0'CB_0$ it follows that $B_0C / 2R = r / BI$, i.e. $2R \cdot r = BI \cdot B_0C$. From the Euler formula it follows that power of I with respect to Ω is $-2R \cdot r = -BI \cdot IB_0$. Therefore $BI \cdot B_0C = BI \cdot IB_0$, it means that in the triangle B_0CI $\angle B_0IC = \angle ICB_0$, but $\angle B_0IC = \angle B/2 + \angle ICB$, $\angle ICB_0 = \angle B/2 + \angle ICA$. We obtain that $\angle IBC = \angle ICA$. It means that the lines AC and BC are symmetric with respect to CI, therefor AC is tangent to ω.

4. Consider the circles $\Omega = (O, R)$ and $\omega_2 = (I, r_2)$, which are the circumcircle and the excircle of some triangle. From problem 2 it yields that $I_2O^2 = R^2 + 2R \cdot r_2$. Take any point B in Ω and let the lines BA and BC be tangents to ω_2 (fig. 4). As the triangles BI_2C' and $B_0'CB_0$

are similar $B_0C/2R = r_2 / BI_2$, i.e. $2R \cdot r_2 = BI_2 \cdot B_0C$, but by Euler formula the degree of point I_2 with respect to Ω is equal to $2R \cdot r_2 = BI_2 \cdot I_2B_0$. So, $BI_2 \cdot B_0C = BI_2 \cdot I_2B_0$. This follows that the triangle B_0CI_2 is isoscelles and $\angle B_0I_2C = \angle I_2CB_0$, but $\angle B_0IC + \angle I_2CB = \angle BB_0C = \angle A$, i.e $\angle I_2CB_0 = \angle A/2$. We obnain that $\angle I_2CA = \angle A/2 + \angle B_0CA = (\angle A + \angle B)/2$. It means that the line I_2C is the external bisector of angle B, therefor AC is tangent to ω_2.

5. Firstly prove that the orthocentric axe is the radical axe of the circumcircle and the nine point circle. Consider two circles: Ω_B with diameter AC and ω_B with diameter HB (fig. 5). The sideline H_1H_3 of orthotriangle is its common chord so lies in its radical axe. Therefor $H_2'H_3 \cdot H_2'H_1 = H_2'A \cdot H_2'C$. Now consider the circumcircle Ω and the nine point circle ω_0. The degrees of point H_2' with respect to Ω and ω_0 are equal to $H_2'A \cdot H_2'C$ and $H_2'H_3 \cdot H_2'H_1$ respectively, i.e the degrees of the common point of respective sidelines of the triangle and its orthotriangle with respect to Ω and ω_0 are equal. This follows that the orthocentric axe is the radical axe of the circumcircle and nine point circle so it is perpendicular to the Euler line.

   Consider now the triangle $I_1I_2I_3$ formed by three excenters. Original triangle ABC is its orthotriangle, and the point I is its orthocenter. So the common points of external bisectors of the triangle ABC with respecive sidelines lie in the orthocentric axe of the triangle $I_1I_2I_3$ i.e in the line perpendicular to the Euler line of this triangle. But the Euler line of the triangle $I_1I_2I_3$ pass through its orthocenter (I) and nine point center (O), therefof it coincide with the line IO.

6. Firstly consider next problem: given two circles $\omega_1 = (O_1, R_1)$ and $\omega_2 = (O_2, R_2)$, their radical axe and center line intersect in the point P (fig. 6). Find the lenght of the segment PO_1. As the degrees of P with respect to both circles are equal, $PO_1^2 - R_1^2 = PO_2^2 - R_2^2$, $PO_2^2 - PO_1^2 = R_2^2 - R_1^2$, $O_1O_2 \cdot (2PO_1 + O_1O_2) = R_2^2 - R_1^2$. So it is easy to express PO_1 through the radius of the circles and the distance O_1O_2..

   Now take the circumcircle of the triangle ABC with radius R as ω_1, and the circle $(I_1I_2I_3)$ with radius $2R$ as ω_2. Then the distance d_1 from the circumcenter O to radical axe (ℓ) is equal to $\frac{R^2 + Rr}{\sqrt{R^2 - 2Rr}}$. Therefor the required distance is equal to

$$d = d_1 - IO = \frac{R^2 + Rr}{\sqrt{R^2 - 2Rr}} - \sqrt{R^2 - 2Rr} = \frac{3Rr}{\sqrt{R^2 - 2Rr}}.$$

7. The solution is analogously to the solution of the problem 5 with replacing of the triangle $I_1I_2I_3$ to the triangles II_2I_3, I_1II_3, I_1I_2I. The circumcircle (Ω) is the common nine-point circle of all these triangles and the lines I_kO are the Euler lines of respective triangles. So the internal bisecors axis of the triangle ABC are the the radical axis of Ω and the circumcircles of the respective triangles.

8. Let Ω and ω_2 be the circumcircle and the excircle of the triangle ABC (fig. 8). Let D be the touching point of its common external tangent with Ω. There are two limit "triangles" in the family of triangles with Ω and ω_2 as the circumcircle and the excircle. Consider a case when the secant AB becomes tangent. This will be the common external tangent to Ω and ω_2. Then the points A and B coincide in the point D, and the lines BC and AC coincide in the tangent PD.
Consider now the circles Ω and ω_2, such that the tangent to ω_2 in its common point P passes through the point D. Then $DK^2 + (r_2 - R)^2 = OI_2^2$, $\text{tg}(\angle I_2DK) = \dfrac{r_1}{DK} = t$. As $\angle PDK = 2 \cdot \angle I_2DK$ we obnain $\sin(\angle PDK) = \dfrac{2 \cdot t}{1+t^2} = \dfrac{2 \cdot r_2 \cdot DK}{DK^2 + r_2^2}$. The lenght of chord DP of circle Ω is equal to $DP = 2R \cdot \sin(\angle DPK)$. But as DP and DK are the tangents to ω_2, $DP = DK$. So $2R\dfrac{2r_2 \cdot DK}{DK^2 + r_2^2} = DP$, $4R \cdot r_2 = DK^2 + r_2^2$. Using the expression for DK^2 we obtain the Euler formula $I_2O^2 = R^2 + 2Rr_2$.

9. As the circles Ω and ω are fixed, and by problem 6 the distance from the center of ω to the external bisectors axe can be expressed through the radius of these circles, we obtain that all feet of external bisectors lies in the fixed line.
Inversely. Let A_2 be an arbitrary point of this line. Let B and C be the common points of tangent to ω passing through A_2 with Ω. By Poncelet theorem the sideline BC generate the triangle ABC which have A_2 as the foot of the external bisector.

10. As the circles Ω and ω_1 are fixed, and by problem 8 the internal bisectors axe ℓ_1 passes through the touching points of of common external tangents of these circles, ℓ_1 is the fixed line.
Inversely. Let B_1 be an arbitrary point in the segment PQ. Let A and C be the common points of tangent to ω_1 passing through B_1 with Ω. By external Poncelet theorem the sideline AC generate the triangle ABC which have B_1 as the foot of the internal bisector.

11. Let be $R = r_2$. By problem 8 this is eqivalent that the common external tangents to Ω and ω_2 are parallel to the line OI_2. So DE is the diameter of the circumcenter i.e. $O \in A_1C_1$.

12. Let ω' be the circle with IB_0' as a diameter. Let ω'' be the circle with II_2 as a diameter (fig.12). The line $B_0'B_2$ is the radical axe of ω' and Ω. The line CB_2 is the radical axe of ω'' and Ω. So the line IB_2 is the radical axe of ω' and ω''. Let K be the second common point of these circles. As $\angle IKB_0' = \angle IKI_2 = 90°$, the point K lies in the line $B_0'I_2$, and therefor $B_0'I_2 \perp B_2I$.