BASIC PLANAR SETS

A. Skopenkov and I. Shnurnikov

problems proposed before the intermediate finish.

Motivation.

In the course of the solution of the Hilbert's 13-th problem the notion of *basic embedding* appeared. The main result of the present sequence of problems (problem 8b) is an elementary solution of 'half' of the Arnold problem on the characterization of basic subsets of the plane. The most important unsolved problems here concern the characterization of *smoothly basic* subsets of the plane.

The more difficult problems are marked by a star, and unsolved problems by two stars. If the statement of a problem is an assertion, then it is required to prove this assertion.

Discontinuously basic subsets.

1. (a) Is it true that for any four numbers $f_{11}, f_{12}, f_{21}, f_{22}$ there exist four numbers g_1, g_2, h_1, h_2 such that $f_{ij} = g_i + h_j$ for each $i, j = 1, 2$?

(b) Andrey Nikolaevich and Vladimir Igorevich play the 'Dare you to decompose!' game. Some cells of chessboard are marked. A. N. writes numbers in the marked cells as he wishes. V. I. looks at the written numbers and chooses (as he wishes) 16 numbers $a_1, \dots, a_8, b_1, \dots, b_8$ as 'weights' of the columns and the lines. If each number in a marked cell turns out to be equal to the sum of weights of the line and the row (of the cell), then V. I. wins, and in the opposite case (i.e., when the number in at least one marked cell is not equal to the sum of weights of the line and the row) A. N. wins.

Prove that V. I. can win no matter how A. N. plays if and only if there does not exist a closed route of a rook starting and turning only at marked cells (the route is not required to pass through each marked cell).

Let $\mathbb{R}^2$ be the plane with a fixed coordinate system. Let $x(a)$ and $y(a)$ be the coordinates of a point $a \in \mathbb{R}^2$. An ordered set (either finite or infinite) $\{a_1, \dots, a_n, \dots\} \subset \mathbb{R}^2$ is called an *array* if for each i we have $a_i \neq a_{i+1}$ and $x(a_i) = x(a_{i+1})$ for even i and $y(a_i) = y(a_{i+1})$ for odd i. It is not assumed that points of an array are distinct. An array is called *closed* if $a_1 = a_{2l+1}$.

2. Consider a closed array $\{a_1, \dots, a_n = a_1\}$. A *decomposition* for such an array is an assignment of numbers at the projections of the points of the array on the x-axis and on the y-axis. Is it possible to put numbers $f_1, \dots, f_n \in \mathbb{R}$, where $f_1 = f_n$, at the points of the array so that for each decomposition there exists an f_i that is not equal to the sum of the two numbers at $x(a_i)$ and $y(a_i)$?

A subset $K \subset \mathbb{R}^2$ is called *discontinuously basic* if for each function $f : K \to \mathbb{R}$ there exist functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

3. (a) The segment $K = 0 \times [0, 1] \subset \mathbb{R}^2$ is discontinuously basic.

(b) The cross $K = 0 \times [-1, 1] \cup [-1, 1] \times 0 \subset \mathbb{R}^2$ is discontinuously basic.

4. (a) *A criterion for a subset of the plane to be discontinuously basic.* A subset of the plane is discontinuously basic if and only if it does not contain any closed arrays.

(b)** Given a set of marked unit cubes in the cube $8 \times 8 \times 8$, how can we see who wins in the 3D analogue of the 'Dare you to decompose!' game? In this analogue V. I. tries to choose 24 numbers $a_1, \dots, a_8, b_1, \dots, b_8, c_1, \dots, c_8$ so that the number at the unit cube (i, j, k) would be equal to the sum $a_i + b_j + c_k$ of the three weights.

(c)** Define discontinuous basic subsets of the 3-space. Discover and prove the 3D analogue of the above criterion.

Continuously basic subsets.

Denote by $|z, z_0| = |(x,y),\ (x_0,y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$ the ordinary distance between points $z=(x,y)$ и $z_0=(x_0,y_0)$ of the plane. Let K be a subset of $\mathbb{R}^2$. A function $f : K \to \mathbb{R}$ is called *continuous* if for each point $z_0 \in K$ and number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each point $z \in K$ if $|z, z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$. It is sometimes convenient to write (x, y) instead of z.

5. (a) The function $f(x,y) = \sqrt{x^2+y^2}$ is continuous on the plane.

(b) The function $f(x,y)$ equal to the integer part of $x+y$ is not continuous on the plane.

(c) Let $a_1,\dots,a_n$ be distinct points of $K \subset \mathbb{R}^2$. Prove that there exists a continuous function $f : K \to \mathbb{R}$ such that $f(a_i) = (-1)^i$ and $|f(x)| \le 1$ for each $x \in K$.

(d) Let $K = \{a_1,\dots,a_{4n+4}\}$ be an array of $4n+4$ distinct points and $f_1,\dots,f_{4n+4}$ be numbers such that $|(-1)^i - f_i| \le \frac{1}{2n}$. Let $g(x(a_i)), h(y(a_i))$, $i = 1,\dots,4n+4$, be numbers such that $f_i = g(x(a_i)) + h(y(a_i))$ for each i. Prove that $\max_i |g(x(a_i))| > n$.

In the sequel all functions are assumed to be continuous.

A subset $K \subset \mathbb{R}^2$ is called *(continuously) basic* if for each continuous function $f : K \to \mathbb{R}$ there exist continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$.

6. (a) A closed array is not basic.

(b) The segment $K = 0 \times [0,1] \subset \mathbb{R}^2$ is basic.

(c) The cross $K = 0 \times [-1,1] \cup [-1,1] \times 0 \subset \mathbb{R}^2$ is basic.

7. (a) If a subset of the plane is basic, then it is discontinuously basic.

(b) A *completed array* is the union of a point $a_0 \in \mathbb{R}^2$ with an infinite array $\{a_1,\dots,a_n,\dots\} \subset \mathbb{R}^2$ of distinct points which *converges* to the point a_0 (i.e. for each $\varepsilon > 0$ there exists a positive integer N such that for each $i > N$ we have $|a_i, a_0| < \varepsilon$). Prove that any completed array is not basic. (Note that it is discontinuously basic).

(c) Let $[a,b]$ be the rectilinear arc which connects points a and b. Prove that the cross $K = [(-1,-2),(1,2)] \cup [(-1,1),(1,-1)]$ is not basic.

(d) Let $x_{ij} = 2 - 3\cdot 2^{-i} + j \cdot 2^{-2i}$. Consider the set of points $(x_{i,2l}, x_{i,2l})$ and $(x_{i,2l}, x_{i,2l-2})$, where i varies from 1 to ∞ and $l = 1,2,3,\dots,2^{i-1}$. Prove that this subset of the plane does not contain any infinite arrays but contains arbitrary long arrays.

(e) The union of the set from the previous problem and the point $(2,2)$ is not basic.

8. Let $K \subset \mathbb{R}^2$ be the image of an arc $[0,1]$ under a continuous map $[0,1] \to \mathbb{R}^2$.

(a) Each continuous function $f : K \to \mathbb{R}$ assumes its lowest value and greatest value. Hint: reduce this problem to an analogous theorem on continuous functions $[0,1] \to \mathbb{R}$.

(b)* If K contains arbitrary long arrays, then K is not basic.

Hint. Assume that K contains arbitrary long arrays and is basic. We may assume that points of each array are distinct. Therefore for each n there is an array $\{a_1^n,\dots,a_{4n+4}^n\}$ of $4n+4$ distinct points in K. Then there exists continuous function $f_n : K \to \mathbb{R}$ such that $f_n(a_i^n) = (-1)^i$ and $|f_n(x)| \le 1$ for each $x \in K$. For each function $G : K \to \mathbb{R}$ its *maximum* is $|G| := \max_{x\in K} |G(x)|$. Let $f : K \to \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$ be functions such that $|f - f_n| < 1/2n$ and $f(x,y) = g(x) + h(y)$ for each $(x,y) \in K$. Then $|g| > n$...

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Problems proposed after the intermediate finish.

The Sternfeld criterion for being a basic subset.

Further hints to problem 8b. Define integers s_n and functions $F_n : K \to \mathbb{R}$ inductively as follows. Set $s_0 = 1$ and $F_0 = 0$. Suppose now that F_{n-1} and s_{n-1} are defined. If F_{n-1} is not representable as $G_{n-1}(x) + H_{n-1}(y)$, then we are done. If it is representable in this way, then take

$$s_n > s_{n-1}!(|G_{n-1}| + n) \quad \text{and} \quad F_n = F_{n-1} + \frac{f_{s_n}}{s_{n-1}!}$$

It remains to prove that if we can construct in this way an infinite number of s_n and F_n, then the function

$$F = \lim_{n\to\infty} F_n = \sum_{n=1}^{\infty} \frac{f_{s_n}}{s_{n-1}!}$$

is not representable as $G(x) + H(y)$.

A sequence of points $\{a_1, \dots, a_n, \dots\} \subset \mathbb{R}^2$ *converges to a point* $a \in \mathbb{R}^2$ if for each $\varepsilon > 0$ there exists an integer N such that for each $i > N$ we have $|a_i, a| < \varepsilon$.

A subset $K \subset \mathbb{R}^2$ of the plane is called *closed*, if for each infinite sequence of points $a_i \in K$ converging to a point a this point belongs to K.

9. (a) A subset $K \subset \mathbb{R}^2$ of the plane is closed if and only if for each point $a \notin K$ there exists $\varepsilon > 0$ such that if for a point b of the plane we have $|a, b| \le \varepsilon$, then b does not belong to K.

(b) The image of an arc under a continuous map $[0, 1] \to \mathbb{R}^2$ is a closed subset of $\mathbb{R}^2$.

The Sternfeld criterion for being a basic subset. A closed bounded subset $K \subset \mathbb{R}^2$ of the plane is basic if and only if K does not contain arbitrary long arrays.

10. (a) The criterion is false without the assumption that K closed.

(b) The criterion is false without the assumption that K bounded.

(c) Prove the 'only if' part ($\Rightarrow$) of the criterion.

Suppose that K is a subset of $\mathbb{R}^2$. For every point $v \in K$ consider the pair of lines passing through v and parallel to the x-axis and the y-axis. If one of these two lines intersects K only at point v, we colour v in white. Define $E(K)$ as the set of noncoloured points of K:

$$E(K) = \{v \in K : \ |K \cap (x = x(v))| \ge 2 \text{ and } |K \cap (y = y(v))| \ge 2\}.$$

Let $E^2(K) = E(E(K))$, $E^3(K) = E(E(E(K)))$ etc.

11. (a) If a subset K of the plane does not contain arbitrary long arrays, then $E^n(K) = \emptyset$ for some n.

(b) Prove the converse statement.

(c)* If K is a closed bounded subset of $\mathbb{R}^2$ and $E(K) = \emptyset$, then K is basic.

(d)* Prove the 'if' part ($\Leftarrow$) of the criterion. Remark. It can be proven first that for piecewise-linear maps f there is a decomposition $f(x, y) = g(x) + h(y)$ with $|g| + |h| < C_n|f|$, where C_n depends only on that n for which $E^n(K) = \emptyset$.

12. (a) Define the property of being a (continuously) basic subset of $\mathbb{R}^3$. Prove that the hedgehog $0 \times 0 \times [-1, 1] \cup 0 \times [-1, 1] \times 0 \cup [-1, 1] \times 0 \times 0 \subset \mathbb{R}^3$ is basic.

(b) The set of 4 points $(0,0,0)$; $(1,1,0)$; $(0,1,1)$; $(1,0,1)$ is basic. (But $E^n(K)\neq\emptyset$ for each n, see below.)

(c)* Let K be a closed bounded subset of $\mathbb{R}^3$. Define $E(K)$ analogously to the above, only instead of lines we use planes orthogonal to the axes:

$$E(K)=\{v\in K:\ |K\cap(x=x(v))|\geq 2,\ |K\cap(y=y(v))|\geq 2 \text{ and } |K\cap(z=z(v))|\geq 2\}.$$

Prove that if $E^n(K)=\emptyset$ for some n, then K is basic.

Smoothly basic subsets of the plane.

Let K be a subset of the plane $\mathbb{R}^2$. A function $f:K\to\mathbb{R}$ is called *differentiable* if for each point $z_0\in K$ there exist a vector $a\in\mathbb{R}^2$ and infinitesimal function $\alpha:\mathbb{R}^2\to\mathbb{R}$ such that for each point $z\in K$

$$f(z)=f(z_0)+a\cdot(z-z_0)+\alpha(z-z_0)|z,z_0|.$$

Here the dot denotes scalar product of vectors $a=(f_x,f_y)$ and $z-z_0=(x,y)$, i.e. $a\cdot(z-z_0)=xf_x+yf_y$. A function $\alpha:\mathbb{R}^2\to\mathbb{R}$ is *infinitesimal*, if for each number $\varepsilon>0$ there exists a number $\delta>0$ such that for each point $(x,y)\in\mathbb{R}^2$

$$\text{if}\quad \sqrt{x^2+y^2}<\delta,\quad \text{then}\quad |\alpha(x,y)|<\varepsilon.$$

A subset $K\subset\mathbb{R}^2$ of the plane is called *differentiably basic* if for each differentiable function $f:K\to\mathbb{R}$ there exist differentiable functions $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ such that $f(x,y)=g(x)+h(y)$ for each point $(x,y)\in K$.

13. (a) (b) (c) Solve the analogues of problem 6 for differentiably basic sets.

14. (a) The graph of the function $y=|x|$, where $x\in[-1,1]$, is differentiably basic.

(b) The broken line whose consecutive vertices are $(-2,0)$, $(-1,1)$, $(0,0)$, $(1,1)$ and $(2,0)$ is not differentiably basic. (Note that it is continuously basic).

(c) The completed array $\{([\frac{n+1}{2}]^{-1/2},[\frac{n}{2}]^{-1/2})\}_{n=2}^{\infty}\cup\{(0,0)\}$ is not differentiably basic. (Note that it is also not continuously basic.)

(d) The completed array $\{(2^{-[\frac{n+1}{2}]},2^{-[\frac{n}{2}]})\}_{n=1}^{\infty}\cup\{(0,0)\}$ is differentiably basic. (Note that it is not continuously basic.)

(e)** Is there a continuous map of arc $[0,1]$ to $\mathbb{R}^2$, whose image is differentiably basic but not continuously basic?

15. (a) The cross $K=[(-1,-2),(1,2)]\cup[(-1,1),(1,-1)]$ is not differentiably basic.

(b) Is the subset $\{(t^2,\frac{t^2}{(1+t)^2})\}_{t\in[-\frac{1}{2},\frac{1}{2}]}$ of the plane differentiably basic?

(c)** Find a criterion for graphs in $\mathbb{R}^2$ to be differentiably basic.

16. Let $r\geq 0$ be an integer and $K\in\mathbb{R}^2$ a subset. A function $f:K\to\mathbb{R}$ is called *r times differentiable* if for each point $z_0\in K$ there exist a polynomial $\overline{f}(z)=\overline{f}(x,y)$ degree at most r of 2 variables x and y and infinitesimal function $\alpha:\mathbb{R}^2\to\mathbb{R}$ such that $f(z)=\overline{f}(z-z_0)+\alpha(z-z_0)|z,z_0|^r$ for each point $z\in K$. (This definition differs from the one generally accepted.)

(a) Functions differentiable zero times are exactly continuous functions, and functions differentiable one time are exactly differentiable functions.

(b) For each positive integer r define the property of being an r times differentiably basic subset of the plane $\mathbb{R}^2$.

(c) For each integer $k\geq 0$ there is a subset of the plane which is r times differentiably basic for $r=0,1\dots k$ but is not r times differentiably basic for each $r>k$.

(d)** Find a criterion for graphs in $\mathbb{R}^2$ to be r times differentiably basic.

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Discontinuously basic subsets.

We use definitions given in problems proposed after the intermediate finish.

1. (a) It is not true. If $f_{ij} = g_i + h_j$ for each $i, j = 1, 2$, then $f_{11} + f_{22} = f_{12} + f_{21}$, but this is false for some numbers f_{ij}.

(b) The statement 'only if' follows from the problem 2. Let us prove the 'if' part by induction on the number of the marked cells. If only one cell is marked then we are done. Let K be the set of centres of the marked cells. The set K does not contain any closed array, therefore $\#E(K) < \#K$. So by the induction hypothesis V. I. can win for $E(K)$. Each cell from $K - E(K)$ is the only marked cell on its line or column, thus V. I. can choose the remaining weights for K.

2. Yes, it is. If every f_i is equal to the sum of two numbers at $x(a_i)$ and $y(a_i)$, then $f_1 - f_2 + f_3 - \dots - f_{n-1} = 0$, but this is false for some numbers f_i.

3. (a) Set $h(y) = f(0, y)$ and $g(x) = 0$.

(b) Set $g(x) = f(x, 0)$ and $h(y) = f(0, y) - f(0, 0)$.

4. (a) The statement 'only if' follows from the problem 2. Let us prove the 'if' part. Consider a function $f : K \to \mathbb{R}$. Our aim is to construct functions g and h so that $f(x, y) = g(x) + h(y)$. Two points $a, b \in K$ are called *equivalent* if there is an array $\{a = a_1, \dots, a_n = b\} \subset K$. Now take an equivalence class $K_1 \subset K$. Define function $g : x(K_1) \to \mathbb{R}$ and $h : y(K_1) \to \mathbb{R}$ in the following way. Take any point $a_1 \in K_1$ and set $g(x(a_1)) = f(a_1)$ and $h(y(a_1)) = 0$. If $\{a_1, a_2, \dots, a_{2l}\}$ is an array, then set

$$h(y(a_{2l})) := f(a_{2l}) - f(a_{2l-1}) + \dots - f(a_1) \quad \text{and} \quad g(x(a_{2l})) := f(a_{2l-1}) - f(a_{2l-2}) + \dots + f(a_1).$$

If $\{a_1, a_2, \dots, a_{2l+1}\}$ is an array, then set

$$g(x(a_{2l+1})) := f(a_{2l+1}) - f(a_{2l}) + \dots + f(a_1)$$

($h(y(a_{2l+1}))$ is already defined). Make this construction for each equivalence class. Then set $g = 0$ and $h = 0$ at all other points of $\mathbb{R}$.

Continuously basic subsets.

5. (a) We can set $\delta = \varepsilon$. Then the statement follows from the triangle inequality: $|f(z) - f(z_0)| \le |z, z_0|$.

(b) For the point $(1, 0)$ and $\varepsilon = \frac{1}{2}$ there is no such δ because $|f(1, 0) - f(1 - \frac{\delta}{2}, 0)| = 1 > \frac{1}{2}$.

(c) First define a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$. Denote $s = \min_{i<j} |a_i, a_j|$. Take n disks with centers a_i and radii $\frac{s}{3}$. Outside of these disks set $f = 0$. Inside the i-th disk take f to be $(-1)^i$ in the center a_i, 0 on the boundary and extend it linearly in the distance to a_i. Then restriction of f to $K \subset \mathbb{R}^2$ is a continuous function $K \to \mathbb{R}$.

(d) We have $|(f_2 - f_3 + f_4 - f_5 + \dots - f_{4n+3}) - (4n+2)| \le \frac{4n+2}{2n} \le 3$. Therefore $g(a_2) - g(a_{4n+3}) \ge (4n + 2) - 3 > 2n$, which implies the required inequality.

6. (a) If an array $A = \{a_1, \dots, a_{2l+1}\}$ is basic, then $f(a_1) - f(a_2) + \dots + f(a_{n-2}) - f(a_{2l}) = 0$. But this is false for some functions f. Cf. problem 2.

(b),(c) Analogously to problems 3a,3b.

7. (a) If the subset is not discontinuously basic, then it contains a closed array. Hence the statement follows by extension of f on the subset and using problem 6a.

(b) Define function f by $f(a_n) = \frac{(-1)^n}{n}$. Suppose that $f(x,y) = g(x) + h(y)$ for some g and h. Then

$$f(a_1) - f(a_2) + f(a_3) - f(a_4) + \cdots - f(a_{2l}) = h(y(a_1)) - h(y(a_{2l})).$$

Since $\lim_{l\to\infty} h(y_{2l})$ exists and equals to $h(y(a_0))$, it follows that $\sum_{i=1}^{2l}(-1)^i f(a_i)$ converges when $l \to \infty$, which is a contradiction.

(c) The cross contains a completed array

$$a_{4k+1} = (\frac{-1}{4^k}, \frac{1}{4^k}),\ a_{4k+2} = (\frac{1}{2\cdot 4^k}, \frac{1}{4^k}),\ a_{4k+3} = (\frac{1}{2\cdot 4^k}, \frac{-1}{2\cdot 4^k}),\ a_{4k+4} = (\frac{-1}{4^{k+1}}, \frac{-1}{2\cdot 4^k})$$

Define a function f on this array using problem 7(b) and then extend it (e.g. piecewise linearly) to the cross. Then there are no functions g and h such that $f(x,y) = g(x) + h(y)$.

(d) For every i the set $(x_{i,2l}, x_{i,2l})_{l=1}^{2^{i-1}} \cup (x_{i,2l}, x_{i,2l-2})_{l=1}^{2^{i-1}}$ is an array of 2^i points.

(e) Define a function f by

$$f((x_{i,2l}, x_{i,2l})) := \frac{1}{2^i} \quad \text{and} \quad f(x_{i,2l}, x_{i,2l-2}) := -\frac{1}{2^i}.$$

If $f(x,y) = g(x) + h(y)$ for some g and h, then for every i using array of points $(x_{i,2l}, x_{i,2l})$ and $(x_{i,2l}, x_{i,2l-2})$, where $l = 1, 2, 3, \dots 2^{i-1}$, we obtain $h(2 - \frac{3}{2^i}) - h(2 - \frac{2}{2^i}) = 1$. This contradicts to the continuity of h.

A function $f : K \in \mathbb{R}$ is called *bounded*, if there exists a number M such that $|f(x)| < M$ for every $x \in K$.

8. *Lemma.* Every continuous function $f : [0,1] \to \mathbb{R}$ is bounded.

Proof. Suppose that for each integer n there exists a point $a_n \in [0,1]$ such that $|f(a_n)| > n$. In the sequence a_n take a subsequence a_{n_i} converging to a point $a \in [0,1]$. Since the function f is continuous, the sequence $f(a_{n_i})$ converges to the $f(a)$ when $i \to \infty$. But $|f(a_{n_i})| > n_i$ and $f(a_{n_i}) \to \infty$. This contradiction implies the Lemma.

(a) Take a map $k : [0,1] \to \mathbb{R}^2$ such that $K = k([0,1])$. Consider a composition $f \circ k : [0,1] \to \mathbb{R}$. Let s be a minimal number for which $f(k(t)) \le s$ for each $t \in [0,1]$. If there were no t such that $f(k(t)) = s$ then continuous function $\frac{1}{s - f(k(t))} : [0,1] \to \mathbb{R}$ would be unbounded. This contradiction shows that f assume its greatest value. Analogously f assume its lowest value.

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The Sternfeld criterion for being a basic subset.

For function $G : K \to \mathbb{R}$ set $|G| := \max_{x\in K} |G(x)|$.

8. (b) Assume that K contains arbitrary long arrays and is basic. We may assume that points of each array are distinct. Therefore for each n there is an array $\{a_1^n, \dots, a_{4n+4}^n\}$ of $4n+4$ distinct points in K. Then there exists continuous function $f_n : K \to \mathbb{R}$ such that $f_n(a_i^n) = (-1)^i$ and $|f_n(x)| \le 1$ for each $x \in K$. Let $f : K \to \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$ be functions such that $|f - f_n| < 1/2n$ and $f(x, y) = g(x) + h(y)$ for each $(x, y) \in K$. Then by the problem 5(d) $|g| > n$. It suffices to assume that $F(x, y) = G(x) + H(y)$ and prove that $|G| > n$ for each n. We have

$$F - F_n = F - F_{n-1} - \frac{f_{s_n}}{s_{n-1}!} = \frac{s_{n-1}!(F - F_{n-1}) - f_{s_n}}{s_{n-1}!}.$$

Set $f = s_{n-1}!(F - F_{n-1})$. Then for $n > 2$ we have $s_n - 1 > s_{n-1}$ and

$$|f - f_{s_n}| = s_{n-1}!|F - F_n| < \frac{1}{(s_n - 1) \cdot s_n} \sum_{k=0}^{\infty} \frac{1}{s_{n+1} \cdot \ldots \cdot s_{n+k}} < \frac{1}{(s_n - 1) \cdot s_n} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2s_n}.$$

We have

$$f = s_{n-1}!(G(x) - G_{n-1}(x)) + s_{n-1}!(H(y) - H_{n-1}(y)).$$

So by the above problem 5(d) it follows that $s_{n-1}!|G - G_{n-1}| > s_n$. Therefore

$$|G| + |G_{n-1}| \ge |G - G_{n-1}| > \frac{s_n}{s_{n-1}!} > |G_{n-1}| + n. \quad \square$$

9. (a) Let us prove the 'only if' part. Let K be a closed subset of the plane. Suppose that for some point $a = (x, y) \notin K$ and for each $\varepsilon = \frac{1}{n} > 0$ there exists a point $a_n \in K$ (at least one) such that $|a, a_n| \le \frac{1}{n}$. The sequence of points $a_n \in K$ converges to the point a, thus $a \in K$. Contradiction.

Now let us prove the 'if' part. Suppose that a sequence a_n converges to a point a and the point $a = (x, y)$ is not in K. There exists $\varepsilon > 0$ such that for every point $a_n \in K$ the distance $|a, a_n| > \varepsilon$. This is a contradiction.

(b) Take a continuous map $f : [0, 1] \to \mathbb{R}^2$. Suppose that a sequence of points $\{a_i\}_{i=1}^{\infty}$ in the image of f converges to a point a. Select a point $t_i \in f^{-1}(a_i)$. Now take a subsequence of points $\{t_{i_k}\}$ of $\{t_i\}$ converging to a point $t_0 \in [0, 1]$. Since the map f is continuous, the subsequence $f(t_{i_k})$ converges to the point $f(t_0)$. Therefore $a = f(t_0)$, so $f([0, 1])$ is closed.

10. (a) Any infinite array A not containing closed arrays and converging to a point $a \notin A$ is basic. This follows because each function defined on A is continuous.

(b) A counterexample is $\{(k, k)\}_{k=1}^{\infty} \cup \{(k, k-1)\}_{k=1}^{\infty}$.

(c) The proof is the same as that of problem 8(b) by the following Lemma.

Lemma. Let K be a closed bounded subset of the plane. Then every continuous function $f : K \to \mathbb{R}$ is bounded.

11. (a) Suppose that $E^n(K) \neq \emptyset$ for each n. For each n take a point $a_0 \in E^n(K)$. Then there exist points $a_{-1}, a_1 \in E^{n-1}(K)$ such that $x(a_{-1}) = x(a_0)$ and $y(a_1) = y(a_0)$. Analogously there exist points $a_{-2}, a_2 \in E^{n-2}(K)$ such that $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$ is an array. Analogously we construct an array of $2n+1$ points in K, which is a contradiction.

(b) Suppose that K contains an array of $2n+1$ points $\{a_{-n}, \dots, a_0, \dots, a_n\}$. Then there is an array of $2n-1$ points $\{a_{-n+1}, \dots, a_{n-1}\}$ in $E(K)$. Analogously $a_0 \in E^n(K)$. Thus if $E^n(K) = \emptyset$, then K does not contain an array of $2n+1$ points.

(c)* See solution of the problem 11(d).

(d)* We present a non-elementary solution based on a reformulation of the property of being a basic subset in terms of *bounded linear operators* in *Banach functional spaces*. Denote by $C(X)$ the space of continuous functions on X with the norm $|f| = \sup\{|f(x)| \ : \ x \in X\}$. We may assume that $K \subset I^2 := [0,1] \times [0,1]$. Define a map (*linear superposition operator*)

$$\varphi\colon C(I) \oplus C(I) \to C(K) \quad \text{by} \quad \varphi(g,h)(x,y) := g(x) + h(y).$$

Clearly, the subset $K \subset I^2$ is basic if and only if φ is surjective, or equivalently, epimorphic.

Denote by $C^*(X)$ the space of *bounded linear functions* $C(X) \to \mathbb{R}$ with the norm $|\mu| = \sup\{|\mu(f)| \ : \ f \in C(X),\ |f| = 1\}$. Denote by $pr_x(a)$ and $pr_y(a)$ the projections of a point $a \in K$ on the coordinate axes.

For a subset $K \subset I^2$ define a map (*dual linear superposition operator*)

$$\varphi^*\colon C^*(K) \to C^*(I) \oplus C^*(I) \quad \text{by} \quad \varphi^*\mu(g,h) := (\mu(g \circ pr_x), \mu(h \circ pr_y)).$$

Since $|\varphi^*\mu| \le 2|\mu|$, it follows that φ^* is bounded. By duality, φ is epimorphic if and only if φ^* is monomorphic.

(We remark that φ^* can be injective but not monomorphic. In other words not only some linear relation on $\operatorname{im}\varphi$ can force it to be strictly less than $C(K)$.)

It is clear that φ^* is monomorphic if and only if *there exist $\varepsilon > 0$ such that $|\varphi^*\mu| > \varepsilon|\mu|$ for each unzero $\mu \in C^*(K)$.*

So it remains to prove that $E^n(K) = \emptyset$ implies the latter condition. We present the proof for $n \in \{1,2\}$. The proof for arbitrary n is analogous. We use the following non-trivial fact: *$C^*(X)$ is the space of σ-additive regular real valued Borel measures on X* (in the sequel we call them simply 'measures'). We have

$$\varphi^*\mu = (\mu_x, \mu_y), \quad \text{where} \quad \mu_x(U) = \mu(pr_x^{-1}U) \quad \text{and} \quad \mu_y(U) = \mu(pr_y^{-1}U) \quad \text{for each Borel set} \quad U \subset I.$$

If $\mu = \mu^+ - \mu^-$ is the decomposition of a measure μ into its positive and negative parts, then $|\mu| = \bar{\mu}(X)$, where $\bar{\mu} = \mu^+ + \mu^-$ is the absolute value of μ.

Let D_x (D_y) be the set of points of K which are not shadowed by some other point of K in x- (y-) direction. Take any measure μ on K of the norm 1.

If $n = 1$, then

$$E(K) = \emptyset, \quad \text{then} \quad D_x \cup D_y = K, \quad \text{so} \quad 1 = \bar{\mu}(K) \le \bar{\mu}(D_x) + \bar{\mu}(D_y).$$

Therefore without loss of generality, $\bar{\mu}(D_x) \ge 1/2$. Since the projection onto the x-axis is injective over D_x, it follows that $|\mu_x| \ge 1/2$, thus the required assertion holds for $\varepsilon = \frac{1}{2}$.

If $n = 2$, then

$$E(E(K)) = \emptyset, \quad \text{hence} \quad D_x \cup D_y = K - E(K), \quad \text{so} \quad E(D_x \cup D_y) = \emptyset.$$

Therefore in the case when $\bar\mu(E(K)) < 3/4$ we have $\bar\mu(D_x \cup D_y) > 1/4$ and without loss of generality $\bar\mu(D_x) > 1/8$. Then as for $n = 1$ we have $|\mu_x| > 1/8$, thus the required assertion holds for $\varepsilon = \frac{1}{8}$.

In the case when $\bar\mu(E(K)) \ge 3/4$ we have $\bar\mu(K - E(K)) \le 1/4$. By the case $n = 1$ above without loss of generality $\bar\mu_x(pr_x(E(K))) \ge \bar\mu(E(K))/2$. Hence $|\mu_x| \ge \frac{1}{2}\cdot\frac{3}{4} - \frac{1}{4} = \frac{1}{8}$, thus the required assertion holds for $\varepsilon = \frac{1}{8}$.

If an embedding $K \subset \mathbb{R}^2$ is basic, then we can prove that φ^* is monomorphic without use of φ as follows. Define a linear operator

$$\Psi\colon C^*(I) \oplus C^*(I) \to C^*(K) \quad \text{by} \quad \Psi(\mu_x, \mu_y)(f) = \mu_x(g) + \mu_y(h),$$

where $g, h \in C(I)$ are such that $g(0) = 0$ and $f(x, y) = g(x) + h(y)$ for $(x, y) \in K$. Clearly, $\Psi\varphi^* = \mathrm{id}$ and Ψ is bounded, hence φ^* is monomorphic.

12. (a) A subset $K \subset \mathbb{R}^3$ is called *(continuously) basic* if for each continuous function $f : K \to \mathbb{R}$ there exist continuous functions $g, h, l : \mathbb{R} \to \mathbb{R}$ such that $f(x, y, z) = g(x)+h(y)+l(z)$ for each point $(x, y, z) \in K$.

For each functuion $f : K \to \mathbb{R}$ on the cross K define $g(x) := f(x, 0, 0)$, $h(y) := f(0, y, 0) - f(0, 0, 0)$ and $l(z) := f(0, 0, z) - f(0, 0, 0)$.

(b) Set $g(0) = f(0, 0, 0)$, $h(0) = 0$, $l(0) = 0$,

$$2g(1) = f(0, 0, 0) + f(1, 1, 0) + f(1, 0, 1) - f(0, 1, 1)$$

$$2h(1) = -f(0, 0, 0) + f(1, 1, 0) - f(1, 0, 1) + f(0, 1, 1))$$

$$\text{and} \quad 2l(1) = -f(0, 0, 0) - f(1, 1, 0) + f(1, 0, 1) + f(0, 1, 1).$$

(c)* Analogously to problem 11(d) [St89, §2, Lemma 23.ii].

Smoothly basic subsets of the plane.

13. (a), (b), (c) Analogously to problems 6(a), 3(a) and 3(b).

14. (a) See solution of 16c.

(b) Suppose the broken line is differentiably basic. Since f is differentiable at points $(-1, 1)$ and $(1, 1)$, the following relations hold for sufficiently small $d > 0$:

$$f(-1 + d, 1 - d) - f(-1, 1) = f_1 d - f_2 d + \alpha_{(-1,1)}(d, -d)|(d, -d)|,$$

$$f(-1 - d, 1 - d) - f(-1, 1) = -f_1 d - f_2 d + \alpha_{(-1,1)}(-d, -d)|(-d, -d)|,$$

$$f(1 + d, 1 - d) - f(1, 1) = f_3 d - f_4 d + \alpha_{(1,1)}(d, -d)|(d, -d)| \quad \text{and}$$

$$f(1 - d, 1 - d) - f(1, 1) = -f_3 d - f_4 d + \alpha_{(1,1)}(-d, -d)|(-d, -d)|.$$

Also we have $f(x, y) = g(x) + h(y)$ and both $g(x)$, $h(y)$ are differentiable. Hence

$$f(-1+d, 1-d) - f(-1, 1) = g(-1+d) - g(-1) + h(1-d) - h(1) = g'(-1)d - h'(1)d + \alpha(d)d \quad \text{and}$$

$$f(-1 - d, 1 - d) - f(-1, 1) = g(-1 - d) - g(-1) + h(1 - d) - h(1) = -g'(-1)d - h'(1)d + \alpha(d)d.$$

Therefore $h'(1) = f_2$ (and $g'(-1) = f_1$). Analogously $h'(1) = f_4$. Thus $h'(1) = f_2 = f_4$. But for function $f(x, y) = xy$ we have $f_4 = 1 \ne f_2 = -1$.

(c) Suppose that this completed array is differentiably basic. Set $a_n = ([\frac{n+1}{2}]^{-1/2}, [\frac{n}{2}]^{-1/2})$, $f(a_n) := \frac{(-1)^n}{n}$, $n = 2, 3, \dots$. If $f(x, y) = g(x) + h(y)$ for some functions $g(x)$ and $h(y)$, then the series $f(a_2) - f(a_3) + f(a_4) - \dots$ converges to $g(1) - g(0)$ (analogously to Problem 7d). This is a contradiction because the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges.

(d) Without loss of generality assume that $f(0, 0) = 0$, then take $g(0) = 0$ and $h(0) = 0$. Set

$$h(2^{-k}) = f(2^{-(k+1)}, 2^{-k}) - f(2^{-(k+1)}, 2^{-(k+1)}) + f(2^{-(k+2)}, 2^{-(k+1)}) - \dots,$$

$$g(2^{-k}) = f(2^{-k}, 2^{-k}) - f(2^{-(k+1)}, 2^{-k}) + f(2^{-(k+1)}, 2^{-(k+1)}) - \dots,$$

where the right-hand sides are sums of alternating series. Now $g(x)$ and $h(y)$ may be extended to differentiable functions $\mathbb{R} \to \mathbb{R}$.

15. (a) Define

$$w(0) = 0, \quad w(4^{-i} + 4^{-3i}) = w(4^{-i}) = 0 \quad \text{and} \quad w(4^{-i} + 4^{-3i-1}) = 2^{3i} \quad \text{for} \quad i = 1, 2, 3, \dots.$$

Extend piecewice-linearly to obtain a function $w : [0, 1] \to \mathbb{R}$. For every $x \in [0, 1]$ define $f(x, -x)$ as the area under the graph of w on $[0, x]$. Define $f(x, y) = 0$ on the rest of the cross.

Suppose that $f(x, y) = g(x) + h(y)$ for some differentiable functions g and h. Without loss of generality we assume that $g(0) = h(0) = 0$. Let us prove that g is not differentiable at $x = 1/4$. (In the same way one can prove that g is not differentiable at $x = 4^{-i}$ for each i.)

Using two infinite arrays starting at points $(\frac{1}{4} + d, -\frac{1}{4} - d)$ and $(\frac{1}{4}, -\frac{1}{4})$ and converging to the point $(0, 0)$ we obtain that

$$g\left(\frac{1}{4} + d\right) - g\left(\frac{1}{4}\right) = f\left(\frac{1}{4} + d, -\frac{1}{4} - d\right) - f\left(\frac{1}{4}, -\frac{1}{4}\right) + f\left(\frac{1}{4^2} + \frac{d}{4}, -\frac{1}{4^2} - \frac{d}{4}\right) - f\left(\frac{1}{4^2}, -\frac{1}{4^2}\right) + \dots$$

For every positive $d < \frac{1}{4}$ there is k such that $4^{-3i} < d/4^{i-1}$ for each $i > k$ and $4^{-3k} \geq d/4^{k-1}$. In particular, $4^{-2k} \geq 4d > 4^{-2(k+1)}$. Then

$$2(g\left(\frac{1}{4} + d\right) - g\left(\frac{1}{4}\right)) > 2^{-3(k+1)} + 2^{-3(k+2)} + 2^{-3(k+3)} \dots > 2^{-3(k+1)} \geq \frac{(4d)^{3/4}}{8}.$$

This contradicts to the differentiability of g at $\frac{1}{4}$.

(b) **Conjecture.** The answer is no, the proof is analogous to that of problem 15(a).

(c)** **Conjecture.** A piecewise-linear graph in $\mathbb{R}^2$ is differentiably basic if and only if it does not contain arbitrary long arrays and for each two singular points a and b we have $x(a) \neq x(b)$ and $y(a) \neq y(b)$. A point $a \in K$ is *singular* if the intersection of K with each disk centered at a is not a rectilinear arc.

16. See an extract of [RZ02] on a separate sheet.

Motivations.

In the course of solution of the Hilbert 13-th problem [Ar58] there appeared the notion of *basic embedding* (we give references to surveys not to original papers). The main result of this sequence of problems (problem 8b) is an elementary solution [MT03] of the 'half' of the Arnold problem [Ar58'] on characterization of basic subsets of the plane [St89]. See also [Vo81, Vo82, Sk95, Ku00, Ku03]. This sequence of problems has only some trivial problems in common with [KS97, KS98]. The most important unsolved problems here are on characterization of *smoothly* *basic* subsets of the plane [RZ02]. We would like to acknowledge V. I. Arnold and S. M. Voronin for useful discussions.

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BASIC PLANAR SETS

A. Skopenkov and I. Shnurnikov (presented by them and V. Gorin)

An extract of [RZ02].

14. (a) Take a differentiable function $f : V \to \mathbb{R}$. Since f is differentiable at $(0,0)$, it follows that there exist $a, b \in \mathbb{R}$ such that

$$f(x,|x|) = f(0,0) + ax + b|x| + o(\sqrt{x^2+|x|^2}), \quad \text{where} \quad x \to 0.$$

Take $h(y) = by$ and $g(x) = f(x,|x|) - h(|x|)$. Clearly, h is differentiable and g is differentiable outside 0. Since $g(x) = f(0,0) + ax + o(x)$ when $x \to 0$, it follows that g is differentiable also at 0.

16. (a) It is clear.

(b) A subset $K \subset \mathbb{R}^2$ is called r *times differentiably basic* if for each r times differentiable function $f : K \to \mathbb{R}$ there exist r times differentiable functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$.

(c) We can take the graph V_k of the function $y = |x|^k$, $x \in [-1,1]$ for k odd, and $W_{k+1} = (V_{k+1} - (2,0)) \cup (V_{k+1} + (2,0))$ for k even.

Proof for k even. Let us prove that W_{k+1} is r times differentiably basic for each $0 \le r \le k$. Given an r times differentiable function $f : W_{k+1} \to \mathbb{R}$, take functions $h(y) = 0$ and $g(x) = f(x, |x - 2\operatorname{sign} x|^{k+1})$. Clearly, h is r times differentiable and $f(x,y) = g(x) + h(y)$ for each $(x,y) \in W_{k+1}$. Since the function $p(t) = |t|^{k+1}$ is k times differentiable and $r \le k$, it follows that g is r times differentiable.

Let us prove that W_{k+1} is not r times differentiably basic for k even and each $k < r$. Define an differentiable function $f : W_{k+1} \to \mathbb{R}$ by $f(x,y) = y \operatorname{sign} x$. If W_{k+1} is r times differentiably basic, then there are r times differentiable functions g and h such that $f(x,y) = g(x) + h(y)$. For $x \in [-1,1]$ we have

$$g(\pm 2 + x) + h(|x|^{k+1}) = f(\pm 2 + x, |x|^{k+1}) = \pm |x|^{k+1}.$$

Since g is $(k+1)$ times differentiable and $k+1$ is odd, it follows that $\frac{dh}{dx}|_{x=0} = +1$ and $\frac{dh}{dx}|_{x=0} = -1$, which is a contradiction. $\square$

Proof for k odd.

Now we prove that V_k is r times differentiably basic for each $0 \le r \le k$. Take an r times differentiable function $f : V_k \to \mathbb{R}$. Since f is r times differentiable at $(0,0)$, it follows that there exist $\{a_{ij}\}_{i,j=0}^r \subset \mathbb{R}$ such that

$$a_{00} = f(0,0) \quad \text{and} \quad f(x,|x|^k) = \sum_{i,j=0}^{r} a_{ij} x^i |x|^{kj} \; + \; o([x^2 + x^{2r}]^{r/2}), \quad \text{where} \quad x \to 0.$$

Since

$$o([x^2 + x^{2r}]^{r/2}) = o_1(x^r), \quad \text{we have} \quad f(x,|x|^k) = a_{00} + a_{01}|x|^k + a_{10}x + \dots + a_{r0}x^r + o_2(x^r).$$

Take $h(y) = a_{01}y$ and $g(x) = f(x,|x|^k) - h(|x|^k)$. Clearly, h is r times differentiable and g is r times differentiable outside 0. We also have $g(x) = a_{00} + a_{10}x + \dots + a_{r0}x^r + o_2(x^r)$ when $x \to 0$. So g is r times differentiable also at 0.

Next we prove that $V = V_1$ is not r times differentiably basic for each $1 < r$. Define a differentiable function $f : V \to \mathbb{R}$ by $f(x,y) = xy$, where $y = |x|$. If V is r times differentiably basic for some $r \geq 2$, then there are r times differentiable functions

$$g, h : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad f(x,|x|) = x|x| = g(x) + h(|x|) \quad \text{for each} \quad x \in [0,1].$$

Hence $g(x) - g(-x) = 2x^2$. But this is impossible because g is 2 times differentiable, hence

$$g(x) = g(0) + ax + bx^2 + o(x^2) \quad \text{and so} \quad g(-x) = g(0) - ax + bx^2 + o(x^2) \quad \text{for} \quad x \to +0.$$

At last we prove that V_k is not r times differentiably basic for k odd and each $k < r$. Define a differentiable function $f : V_k \to \mathbb{R}$ by $f(x,y) = xy$, where $y = |x|^k$. If V is r times differentiably basic for some $r > k$, then there are r times differentiable functions

$$g, h : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad f(x,|x|^k) = x|x|^k = g(x) + h(|x|^k) \quad \text{for each} \quad x \in [0,1].$$

Hence $g(x) - g(-x) = 2x|x|^k$. But this is impossible for k odd because g is $(k+1)$ times differentiable, hence

$$g(x) = g_0 + g_1 x + \dots + g_{k+1} x^{k+1} + o(x^{k+1}) \quad \text{and so} \quad g(-x) = g_0 - g_1 x + \dots + g_{k+1} x^{k+1} + o(x^{k+1})$$

$$\text{for} \quad x \to +0. \qquad \square$$