Theorem on altitudes and the Jacobi identity

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Solutions of the problems suggested before the intermediate finish.

1. Denote by O the center of our sphere. Let γ be the plane passing through the points O, A and B. Let c be the spherical line obtained as the intersection of the sphere and γ. Then c is precisely the spherical line passing through the points A and B.

On the other hand, by our definition, the vectors $\vec{A} = \overrightarrow{OA}$ and $\vec{B} = \overrightarrow{OB}$ correspond to the points A and B, respectively, and the vector $[\vec{A}, \vec{B}]$ is orthogonal to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Therefore the vector $[\vec{A}, \vec{B}]$ is also orthogonal to the plane γ. Finally, by our definition of correspondence between vectors and spherical lines, we obtain that the vector $[\vec{A}, \vec{B}]$ corresponds to the spherical line c.

2. Recall that the intersection of two spherical lines is a pair of two diametrically opposite points on the sphere. Let C be one of the intersection points of the spherical lines a and b. We shall now show that the vector $\overrightarrow{OC}$ is parallel to the vector $[\vec{a}, \vec{b}]$. Let α be the plane containing the point O and the spherical line a. By definition of the correspondence between spherical lines and vectors, the vector a is orthogonal to the plane α. The segment OC lies in the plane α, therefore $\overrightarrow{OC} \perp \vec{a}$. Similarly, $\overrightarrow{OC} \perp \vec{b}$. Therefore the vectors $\overrightarrow{OC}$ and $[\vec{a}, \vec{b}]$ are collinear, whence the vector $[\vec{a}, \vec{b}]$ corresponds to the point C (recall here that all vectors collinear to $\overrightarrow{OC}$ correspond to the point C).

3. Let c be the line passing through the point A and orthogonal to the line b. We shall say that c is the *perpendicular* dropped from A onto b. It suffices to prove that the vector $\vec{c}$ corresponding to the line c is orthogonal to both the vector $\vec{A}$ and the vector $\vec{b}$ (indeed, in this case the vector $\vec{c}$ is collinear to the vector $[\vec{A}, \vec{b}]$ and we know that collinear vectors correspond to the same line).

First we show that $\vec{c} \perp \vec{A}$. Consider the plane γ passing through the point O and containing the line c. Since the point A belongs to c, the point A also belongs to γ. Therefore $\vec{c}$ is orthogonal to $\overrightarrow{OA} = \vec{A}$.

Now we show that $\vec{c} \perp \vec{b}$. Denote by β the plane passing through the point O and containing the line b. Since spherical lines b and c are orthogonal, the planes β and γ are also orthogonal. But then any vector orthogonal to β must also be orthogonal to any vector orthogonal to γ, and our proof is complete.

4. *Answer:* the points A, B and C lie on the same line.

Solution. The condition $A + B + C = 0$ implies that the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are parallel to some plane π. We may assume here that the plane π passes through the point O and therefore defines a spherical line p. Then the points A, B, and C lie on the spherical line p.

5. *Answer:* the three lines a, b and c all pass through the same point.

Solution. The condition $\vec{a} + \vec{b} + \vec{c} = 0$ implies that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are parallel to some plane π. Take the point P on our sphere such that $\overrightarrow{OP} \perp \pi$. Then the lines a, b and c all pass through P. Indeed, since $\overrightarrow{OP}$ is orthogonal to π, it is also orthogonal to the vector $\vec{a}$, which, by Problem 3, implies that the point P lies on the line a. Similarly, the point P lies on the lines b and c.

6. Consider the identity $[\vec{A}, [\vec{B}, \vec{C}]] = \vec{B}(A, C) - \vec{C}(A, B)$. If to the vector $\vec{A}$ one adds an arbitrary vector collinear to $[\vec{B}, \vec{C}]$ then neither the left nor the right part of the identity changes. Therefore it suffices to consider the case when the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are all parallel to the same plane.

Now observe that neither part of our identity changes if to the vector $\vec{B}$ one adds a vector collinear to $\vec{C}$. Since $\vec{A}$, $\vec{B}$ and $\vec{C}$ are parallel to the same plane, we may assume that either $\vec{B}$ or $\vec{C}$ is collinear to $\vec{A}$.

For definiteness, assume that $\vec{B} \parallel \vec{A}$. Adding to the vector $\vec{C}$ a vector collinear to $\vec{B}$ we may assume that $\vec{C} \perp \vec{B}$. But if $\vec{B} \parallel \vec{A}$ and $\vec{B} \perp \vec{C}$, then it is easy to check directly that both parts of our identity are equal to $|\vec{A}| \cdot |\vec{B}| \cdot \vec{C}$.

Remark. Our identity may also be proven by using linearity of both its parts.

7. The Jacobi identity is obtained by cyclically permuting the variables in the identity of Problem 6 and summing the resulting identities.