FUNCTIONAL EQUATIONS

First stage

A) **A functional equation** is an equation including one or more unknown functions (having prescribed domain and range). To solve a functional equation means to find all functions which satisfy it identically. Functional equations arise in various areas of mathematics, usually when we have to describe all functions having given properties.

We start with some typical methods for solving functional equations. **The method of substitution** is often useful. It consists in replacing variables by some new functions (maybe constants) in order to reduce the equation to some more handy form.

1. Solve the following functional equations.

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x. \tag{a}$$

$$f(x) + f\left(\frac{x-1}{x}\right) = 2x. \tag{b}$$

$$f(x+y) + f(x-y) = 2f(x)\cos\ y. \tag{c}$$

B) Now we shall consider some specific kinds of functional equations. Many functional equations include some iteration of the unknown function. The following Problem 2 is said to be often given by Feynman to his young colleagues.

2. Does there exist a function $f(x) : R \to R$ such that $f(f(x)) = x^2 - 2$ for all real x?

3. Find all functions $f : R^2 \to R$ which satisfy the functional equation

$$f(\dots f(f(x_1, x_2), x_3) \dots, x_{2006}) = x_1 + x_2 + x_3 + \dots + x_{2006}.$$

C) The result of solving a functional equation often depends on whether we impose the requirement of continuity on the functions sought. We shall not need the strict definition of continuity here. It suffices to know that any polynomial, exponent, logarithm (for $x > 0$), sine, cosine are continuous, and that any continuous function has the following properties.

(a) If the values of a continuous function at the points a and b are different then any intermediate value is achieved in some point of the interval $[a; b]$ (the Intermediate Value Theorem).

(b) If two continuous functions defined on the real axis coincide at all rational points then they coincide everywhere.

(c) If a continuous function is 1-1 then it is strictly monotonic. (The inverse is obvious.)

(d) A function continuous on a closed interval $[a; b]$ is bounded on this interval.

Monotonicity of a function also is sometimes important for solving functional equations.

4. If a function $f(x)$ is strictly monotonic, what can you say about growth and decrease of the function $f(f(x))$?

5. Does there exist a continuous function $f : R \to R$ such that $f(f(x))$ strictly decreases?

6. A continuous function $f(x)$ is such that $f(f(x)) = -x^2$ for all real x. Prove that $f(x) \le 0$ for all real x.

Supplementary questions for problem 6.
(a) Does there exist any function satisfying the conditions of the problem?
(b) Is the continuity condition essential here?
7. Does there exist a continuous function $f(x) : R \to R$ such that $f(f(x)) = x^2 - 1/2$ for all real x? (Compare with Problem 2.)

D) Now we shall consider the most famous functional equation, namely the **additive Cauchy equation**, often simply called the **Cauchy equation**:

$$f(x + y) = f(x) + f(y) \quad (x, y \in R).$$

8. Find all continuous solutions for the additive Cauchy equation.
Which of the properties (a)-(d) of continuous functions has been used here? The method of finding continuous solutions of functional equations using this property is called the **Cauchy method**.
As is well known, $(xy)^n = x^n y^n$, $\exp(x + y) = \exp(x)\exp(y)$ $(x, y \in R)$ and $\ln(|xy|) = \ln(|x|) + \ln(|y|)$ $(x, y \in R \setminus \{0\})$. Using this facts and the result of Problem 8, solve the following problem.
9. Find all continuous solutions for the **Cauchy equations**:

$$f(xy) = f(x) + f(y) \quad (x, y \in R \setminus \{0\}); \tag{a}$$

$$f(x + y) = f(xy) \quad (x, y \in R); \tag{b}$$

$$f(xy) = f(x)f(y) \quad (x, y \in R). \tag{c}$$

A Hamel basis is a set of real numbers such that any real number has a unique representation of the form $r_1\alpha_1 + \ldots r_n\alpha_n$ where n is a positive integer, $r_1, \ldots, r_n$ are rationals, and $\alpha_1, \ldots, \alpha_n$ belong to the given Hamel basis. The existence of a Hamel basis is proved using the axiom of choice, and here we assume it as a given fact.
10. Do there exist any discontinuous solutions of the additive Cauchy equation? If yes, then how can the set of all its solutions be described? How can one describe the solutions of this equation that are non-negative for $x \geq 0$?
For comparison, now solve the following problem.
11.

$$f(x + y) = f^n(x) + f^n(y) \ (x, y \in R;\ n \text{ is a fixed positive integer, } n > 1).$$

E) The **Pexider equation** is obtained from the additive Cauchy equation by replacing all occurences of f with different functions:

$$k(x + y) = g(x) + h(y).$$

12. (a) Solve the Pexider equation.
(b) Find all its continuous solutions.